SPECTRAL STABILITY OF THE $\bar{\partial}$–NEUMANN LAPLACIAN:
THE KOHN-NIRENBERG ELLIPTIC REGULARIZATION

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Abstract. In this paper we study spectral stability of the $\bar{\partial}$-Neumann Laplacian under
the Kohn-Nirenberg elliptic regularization. We obtain quantitative estimates for stability
of the spectrum of the $\bar{\partial}$-Neumann Laplacian when either the operator or the underlying
domain is perturbed.

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ational eigenvalue, pseudoconvex domain, finite type condition.

1. Introduction

The $\bar{\partial}$-Neumann Laplacian $\Box_q$ on a bounded domain $\Omega$ in $\mathbb{C}^n$ is (a constant multi-
ple of) the usual Laplace operator acting diagonally on $(p,q)$-forms with the $\bar{\partial}$-Neumann
boundary condition. It is the archetype of an elliptic operator with non-coercive bound-
dary condition. Subelliptic estimate for the $\bar{\partial}$-Neumann Laplacian on smoothly bounded
strongly pseudoconvex domains in $\mathbb{C}^n$ was established by Kohn [Ko63] (see [DK99] for an
exposition on related subjects). One difficulty in studying non-coercive boundary value
problems is to show that a priori estimates of derivatives imply that these derivatives
exist and the same estimates hold without prior regularity assumptions. Elliptic regu-
larization was introduced by Kohn and Nirenberg [KN65] to resolve this difficulty. By
adding a positive constant $t$ multiple of an elliptic operator to the $\bar{\partial}$-Neumann Laplacian,
the $\bar{\partial}$-Neumann problem is converted into a coercive elliptic problem for which existence
of the derivatives is well known. One then obtains bona fide estimates from a priori ones
by taking $t \to 0^+$, provided the desired estimates are uniform in $t$.

Spectral stability for the classical Dirichlet and Neumann Laplacians on domains in $\mathbb{R}^n$
has been studied extensively in the literatures (see, e.g., [F99, D00, BL08] and references
therein). Less is known of spectral stability for the $\bar{\partial}$-Neumann Laplacian. In [FZ19], we
studied spectral stability of the $\bar{\partial}$-Neumann Laplacian $\Box_q$ on a bounded domain $\Omega$ in $\mathbb{C}^n$
as the underlying domain is perturbed. We established upper semi-continuity properties
for the variational eigenvalues of the $\bar{\partial}$-Neumann Laplacian on bounded pseudoconvex
domains, lower semi-continuity properties on pseudoconvex domains that satisfy Catlin’s
property ($P$), and quantitative estimates on smoothly bounded pseudoconvex domains
of finite type in the sense of D’Angelo. In this paper, we consider the perturbation $\Box_q^t$
of the $\bar{\partial}$-Neumann Laplacian introduced by Kohn and Nirenberg [KN65] in their elliptic

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regularization procedure. We study stability of the spectrum of \(\square_t\), first as \(t \to 0^+\) and then as the underlying domain \(\Omega\) is perturbed.

Unlike the classical Dirichlet or Neumann Laplacian, the spectrum of the \(\bar{\partial}\)-Neumann Laplacian need not be purely discrete (see [FS01] for an exposition on the subject). There are several ways to measure spectral stability under this circumstance. Here our focus is on stability of the variational eigenvalues defined by the Min-Max Principle and convergence of the operators in resolvent sense (see Section 2 below for the precise definitions). When the spectrum is purely discrete, the variational eigenvalues are indeed eigenvalues, arranged in increasing order and repeated according to multiplicity. Let \(\lambda_k^q(\Omega)\) be the \(k\)-th variational eigenvalues of the \(\bar{\partial}\)-Neumann Laplacian \(\square_q\) on \((0, q)\)-forms, \(1 \leq q \leq n - 1\), on \(\Omega\). Let \(\lambda_k^{t,q}(\Omega)\) be the \(k\)-th-eigenvalue of \(\square_t^q\). Our first result concerns spectral stability of \(\square_t^q\) as \(t \to 0^+\) (see Theorem 3.2 and Theorem 3.3 in Section 3):

**Theorem 1.1.** Let \(\Omega\) be a bounded domain in \(\mathbb{C}^n\) with \(C^2\)-smooth boundary. Let \(1 \leq q \leq n - 1\) and let \(k \in \mathbb{N}\). Then \(\square_t^q\) converges to \(\square_q^q\) in strong resolvent sense as \(t \to 0^+\) and

\[
\lim_{t \to 0^+} \lambda_k^{t,q}(\Omega) = \lambda_k^q(\Omega).
\]

Furthermore, if \(\Omega\) is strongly pseudoconvex with smooth boundary, then \(\square_t^q\) converges to \(\square_q^q\) in norm resolvent sense and if \(\Omega\) is pseudoconvex of finite type in the sense of D’Angelo, then there exist positive constants \(\alpha \in (0, 1/2]\) and \(C\) independent of \(t\) and \(k\) such that

\[
|\lambda_k^{t,q}(\Omega) - \lambda_k^q(\Omega)| \leq Ctk(\lambda_k^q(\Omega))^{2([1/2\alpha]+1)},
\]

where \([1/2\alpha]\) is the integer part of \(1/2\alpha\).

Our next result is about spectral stability of the Kohn-Nirenberg elliptic regularization operator \(\square_t^q\) as the underlying domain \(\Omega\) is perturbed. Perturbation of the domain is measured in the \(C^2\)-topology. Our main result in this direction is the following quantitative estimate:

**Theorem 1.2.** Let \(\Omega\) and \(\Omega_j\) be smooth bounded pseudoconvex domains in \(\mathbb{C}^n\) with normalized defining functions \(r\) and \(r_j\) respectively. Assume that \(C^\infty\)-norms of \(r_j\) are uniformly bounded on \(\overline{\Omega}_j\). Let \(\delta_j = \|r - r_j\|_{C^2}\) be the \(C^2\)-norm over \(\overline{\Omega} \cup \overline{\Omega}_j\). Let \(1 \leq q \leq n - 1\), \(0 < t < 1\) and \(k \in \mathbb{N}\). Then there exist positive constants \(\delta\) and \(C_k\) such that

\[
|\lambda_k^{t,q}(\Omega_j) - \lambda_k^q(\Omega)| \leq \frac{C_k\delta_j}{t^{2n+3-1}},
\]

provided \(\delta_j < \delta\).

This paper is organized as follows. In Section 2, we recall the spectral theoretic setup of the \(\bar{\partial}\)-Neumann Laplacian \(\square_q\) and the Kohn-Nirenberg elliptic regularization \(\square_t^q\). In Section 3, we study spectral stability of \(\square_t^q\) as \(t \to 0^+\) and prove Theorem 1.1. In Section 4, we study spectral stability of \(\square_q^q\) as the underlying domain is perturbed and prove Theorem 1.2. Throughout this paper, we will use \(C\) to denote constants which might not be the same in different appearances.

2. Preliminary

In this section, we review the setup for the \(\bar{\partial}\)-Neumann Laplacian (cf. [FK72, CS99]) and the elliptic regularization of the \(\bar{\partial}\)-Neumann Laplacian([KN65], see also [T96, S10]). We define them through their associated quadratic form.
Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) and let \( L^2_{(0,q)}(\Omega) \) be the space of \((0,q)\)-forms with \( L^2 \)-coefficients on \( \Omega \) with respect to the standard Euclidean metric. Let \( \partial_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega) \) be the maximally defined Cauchy-Riemann operator. The domain \( \text{Dom}(\partial_q) \) of \( \partial_q \) consists of forms \( u \in L^2_{(0,q)}(\Omega) \) such that \( \overline{\partial}_q u \in L^2_{(0,q)}(\Omega) \) in the sense of distribution.

Let \( \overline{\partial}_q^* : L^2_{(0,q+1)}(\Omega) \to L^2_{(0,q)}(\Omega) \) be the adjoint of \( \overline{\partial}_q \). Its domain is then given by

\[
\text{Dom}(\overline{\partial}_q^*) = \{ u \in L^2_{(0,q+1)}(\Omega) \mid \exists C > 0, |\langle u, \overline{\partial}_q v \rangle| \leq C\|v\|, \forall v \in \text{Dom}(\overline{\partial}_q) \}. 
\]

Suppose \( \Omega \) has \( C^1 \)-smooth boundary. Let

\[
u = \sum_{|J|=q} u_J d\bar{z}_J \in C^1_{(0,q)}(\overline{\Omega}),
\]

where the sum takes over all strictly increasing \( q \)-tuple of integers between 1 and \( n \). Then \( u \in \text{Dom}(\overline{\partial}_q^*_{-1}) \) if and only if

\[
(\overline{\partial}_r)^* u = \sum_{|K|=q-1} \left( \sum_{k=1}^n u_{kK} \frac{\partial r}{\partial z_k} \right) d\bar{z}_K = 0
\]
on \( \partial \Omega \), where \( r \) is a defining function of \( \partial \Omega \) such that \( |\nabla r| = 1 \) on \( \partial \Omega \) and

\[
(\overline{\partial}_r)^* = \sum_{j=1}^n \frac{\partial r}{\partial z_j} \frac{\partial}{\partial \bar{z}_j}
\]
is the dual \((0,1)\)-vector field of \( \overline{\partial}_r \) and \( \cdot \) denotes the contraction operator. We decompose \( u = u^\tau + u^\nu \) into the tangential and normal parts where \( u^\nu = (\overline{\partial}_r)^* u \land \overline{\partial}_r \) and \( u^\tau = u - u^\nu \).

For \( 1 \leq q \leq n-1 \), let

\[
Q_q(u, v) = \langle \overline{\partial}_q u, \overline{\partial}_q v \rangle_{\Omega} + \langle \overline{\partial}_q^*_{-1} u, \overline{\partial}_q^*_{-1} v \rangle_{\Omega}
\]
be the sesquilinear form on \( L^2_{(0,q)}(\Omega) \) with domain \( \text{Dom}(Q_q) = \text{Dom}(\overline{\partial}_q) \cap \text{Dom}(\overline{\partial}_q^*) \).

The \( \overline{\partial} \)-Neumann Laplacian \( \Box_q \) is the unique nonnegative self-adjoint operator \( \Box_q \) such that \( Q_q(u, v) = \langle \Box_q^{1/2} u, \Box_q^{1/2} v \rangle_{\Omega} \) with \( \text{Dom}(\Box_q^{1/2}) = \text{Dom}(Q_q) \). Consequently, \( \Box_q \) is given by

\[
\Box_q = \partial_q \overline{\partial}_q^*_{-1} + \overline{\partial}_q^* \partial_q
\]
and

\[
\text{Dom}(\Box_q) = \{ u \in L^2_{(0,q)}(\Omega) \mid u \in \text{Dom}(Q_q), \overline{\partial}_q u \in \text{Dom}(\overline{\partial}_q^*), \overline{\partial}_q^* u \in \text{Dom}(\partial_q) \}. \]

When \( \Omega \) is pseudoconvex, then it follows from Hörmander’s \( L^2 \)-estimates for the \( \overline{\partial} \)-equation that \( \Box_q \) has a bounded inverse \( N_q = \Box_q^{-1} : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega) \), the \( \overline{\partial} \)-Green’s operator ([H65]; see also [CS99]).

We now review the elliptic regularization in the setting of the \( \overline{\partial} \)-Neumann problem ([KN65]). Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \). For \( t > 0 \), let

\[
Q_q(tu, v) = Q_q(u, v) + t\langle \nabla u, \nabla v \rangle_{\Omega}
\]
with \( \text{Dom}(Q_q) = W^1_{(0,q)}(\Omega) \cap \text{Dom}(\overline{\partial}_q^*_{-1}) \), where the gradient operator \( \nabla \) acts componentwise. (Hereafter, we use \( W^s_{(0,q)}(\Omega) \) to denote the space of \((0,q)\)-forms with coefficients
in the $L^2$-Sobolev space of order $s$. The associated norm is denoted by either $\| \cdot \|_{W^s}$ or $\| \cdot \|_s$.) Then $Q_q^{t}$ is a densely defined, closed sesquilinear form on $L^2_{(0,q)}(\Omega)$. Let $\square_q^t$ be the self-adjoint operator associated with $Q_q^{t}$. This is an elliptic operator with coercive boundary condition. It was introduced by Kohn and Nirenberg ([KN65]) to study non-coercive boundary problems such as the $\overline{\partial}$-Neumann problem. For abbreviation, we will call the operator $\square_q^t$ the Kohn-Nirenberg Laplacian.

When $\partial \Omega$ is $C^2$-smooth, then a form $u \in C^2_{(0,q)}(\bar{\Omega})$ belongs to $\text{Dom}(\square_q^t)$ if and only if $u \in \text{Dom}(\partial^*_q-1)$ and

$$(\partial r)^* \partial_q u + t \left( \frac{\partial u}{\partial \nu} \right)^\tau = 0$$

on $\partial \Omega$, where $r$ is a $C^2$-smooth normalized defining function of $\Omega$ and $\left( \frac{\partial u}{\partial \nu} \right)^\tau$ is the tangent part of the (component-wise) normal derivative $\frac{\partial u}{\partial \nu}$ of $u$ (see [S10, § 3.3] and [T96, Ch. 12]).

Let $\lambda^q_k(\Omega)$ and $\lambda^{t,q}_k(\Omega)$ be the $k$th-variational eigenvalues of $\square_q$ and $\square_q^t$ on $\Omega$ respectively, given by the Min-Max Principle as follows:

$$\lambda^q_k(\Omega) = \inf_{L \subset \text{Dom}(Q_q)} \sup_{u \in L \setminus \{0\}} Q_q(u,u)/\|u\|^2$$

and

$$\lambda^{t,q}_k(\Omega) = \inf_{L \subset \text{Dom}(Q_q^{t})} \sup_{u \in L \setminus \{0\}} Q_q^{t}(u,u)/\|u\|^2,$$

where the infima take over all linear $k$-dimension subspaces of $\text{Dom}(Q_q)$ and $\text{Dom}(Q_q^{t})$ respectively. Recall that the spectrum of a non-negative self-adjoint operator $S$ is purely discrete if and only if the variational eigenvalues $\lambda_k(S)$ defined as above goes to $\infty$ as $k \to \infty$. In this case, $\lambda_k(S)$ is the $k$th-eigenvalue of $S$ when the eigenvalues are arranged in increasing order and repeated according to multiplicity (see [D95, Chapter 4]). We collect some elementary properties of the Kohn-Nirenberg Laplacian in the following proposition:

**Proposition 2.1.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$. Let $k$ be a positive integer and let $K = k \cdot n!/q!(n-q)!$. Then

$$\lambda^q_k(\Omega) \leq \lambda^{t,q}_k(\Omega)$$

and

$$t \lambda^N_k(\Omega) \leq \lambda^{t,q}_k(\Omega) \leq \left( \frac{1}{4} + t \right) \lambda^D_k(\Omega),$$

where $\lambda^N_k(\Omega)$ and $\lambda^D_k(\Omega)$ are respectively the $k$th variational eigenvalues of the Neumann and Dirichlet Laplacians. Furthermore, if $\partial \Omega$ is $C^1$-smooth, then $\square_q^t$ has purely discrete spectrum and its first eigenvalue satisfies

$$\lambda^{t,q}_1(\Omega) \geq C \min \{t, 1\},$$
for some constant \( C > 0 \) independent of \( t \). As a consequence, \( N^t_q = (\Box^t_q)^{-1} : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega) \) is compact and satisfies

\[
(2.8) \quad \|N^t_q u\| \leq 1/(C \min\{t, 1\})\|u\|, \quad u \in L^2_{(0,q)}(\Omega).
\]

**Proof.** The inequality (2.5) is a consequence of the Min-Max Principle in the definition of the variational eigenvalues and the fact that

\[
\text{Dom}(Q^t_q) \subset \text{Dom}(Q_q) \quad \text{and} \quad Q_q(u, u) \leq Q^t_q(u, u), \quad u \in \text{Dom}(Q^t_q).
\]

Since

\[
\text{Dom}(Q^t_q) \subset W^1_{(0,q)}(\Omega) \quad \text{and} \quad t\|\nabla u\|^2 \leq Q^t_q(u, u), \quad u \in \text{Dom}(Q^t_q),
\]

we have

\[
\inf_{\{L \in W^1_{(0,q)}(\Omega) \mid \text{dim } L = K\}} \sup_{u \in L}(u \{0\}) \|\nabla u\|^2/\|u\|^2 \leq \frac{1}{t} \inf_{\{L \in \text{Dom}(Q^t_q) \mid \text{dim } L = K\}} \sup_{u \in L\{0\}} Q^t_q(u, u)/\|u\|^2 = \frac{1}{t} \lambda^{q,t}_K(\Omega).
\]

The quantity on the left-hand side is the \( K \)-th variational eigenvalues of the Neumann Laplacian acting componentwise on \((0,q)\)-forms. Recall that

\[
(2.9) \quad \inf_{\{L \in W^1_{(0,q)}(\Omega) \mid \text{dim } L = K\}} \sup_{u \in L}(u \{0\}) \|\nabla u\|^2/\|u\|^2 = \sup_{u \in W^1_{(0,q)}(\Omega) \{0\}} \inf_{u \perp L} \|\nabla u\|^2/\|u\|^2
\]

(compare [D95, Sect. 4.5]). To establish the first inequality in (2.6), it suffices to prove that

\[
\lambda^N_k(\Omega) = \sup_{\{L \in W^1(\Omega) \mid \text{dim } L = k-1\}} \inf_{u \in W^1(\Omega) \{0\}} \|\nabla u\|^2/\|u\|^2 \leq \sup_{\{L \in W^1_{(0,q)}(\Omega) \mid \text{dim } L = k-1\}} \inf_{u \in W^1_{(0,q)}(\Omega) \{0\}} \|\nabla u\|^2/\|u\|^2.
\]

For any \( 0 < \varepsilon < 1 \), there exists a \((k-1)\)-dimensional subspace \( L_{k-1} \subset W^1(\Omega) \) such that

\[
\inf \{ \|\nabla f\|^2 \mid f \in W^1(\Omega), f \perp L_{k-1}, \|f\| = 1 \} > \lambda^N_k(\Omega) - \varepsilon.
\]

Let \( \{u_1, \cdots, u_{k-1}\} \) be an orthonormal basis of \( L_{k-1} \) such that \( \langle \nabla u_i, \nabla u_j \rangle = \gamma_{ij}\delta_{ij}, \quad 1 \leq h, i \leq k-1, \quad 0 \leq \gamma_1 \leq \cdots \leq \gamma_{k-1} \leq \lambda^N_k(\Omega) - \varepsilon \). Let \( L_{k,q} = \{u_jd^q_j \mid 1 \leq j \leq k-1, |J| = q\} \). Since \((0,q)\)-form in \( \mathbb{C}^n \) has \( n!/q!(n-q)! \) components, it follows that \( L_{k,q} \) is a subspace of \( W^1_{(0,q)}(\Omega) \) of dimension \((k-1)n!/q!(n-q)!\) and

\[
\inf_{u \in W^1_{(0,q)}(\Omega) \{0\} \mid u \perp L_{k,q}} \|\nabla u\|^2/\|u\|^2 > \lambda^N_k(\Omega) - \varepsilon.
\]

This concludes the proof of the first inequality of (2.6). The second inequality follows similarly from the fact that

\[
W^1_{0,(0,q)}(\Omega) \subset \text{Dom}(Q^t_q)
\]

and

\[
Q^t_q(u, u) = \left(\frac{1}{4} + t\right)\|\nabla u\|^2, \quad u \in W^1_{0,(0,q)}(\Omega),
\]

where \( W^1_{0,(0,q)}(\Omega) \) is the completion of the space of smooth, compactly supported \((0,q)\)-forms on \( \Omega \) in \( W^1_{(0,q)}(\Omega) \). In this case, when comparing the eigenvalues, one uses the left-hand side of (2.9) to construct the appropriate subspace of \( W^1_{(0,q)}(\Omega) \).
When $\Omega$ has $C^1$-smooth boundary, $W^1_{(0,q)}(\Omega)$ is relatively compact in $L^2_{(0,q)}(\Omega)$. It follows that \{u \in \text{Dom}(Q^t_q) \mid \|u\|^2 + Q^t_q(u, u) \leq 1\} is a relatively compact subset of $L^2_{(0,q)}(\Omega)$. Thus $\Box^t_q$ has compact resolvent and its spectrum is purely discrete. The smallest eigenvalue $\lambda_1^{t,q}(\Omega)$ of $\Box^t_q$ must be positive. Otherwise, if $\lambda_1^{t,q}(\Omega) = 0$, then the corresponding eigenform $u$ satisfies $\|\nabla u\| = 0$ and the $\overline{\partial}$-Neumann boundary condition $u \in \text{Dom}(\overline{\partial}^*)$. Therefore $u$ has constant coefficients. Since $\partial \Omega$ is $C^1$-smooth, there are points on the boundary where only one of the partial derivatives $\partial \rho/\partial z_j$, $1 \leq j \leq n$, of a defining function $\rho$ of $\Omega$ is non-zero. (One can consider, for example, the points furthest from a coordinate hyperplane.) By applying the $\overline{\partial}$-Neumann boundary condition to $u$ on these points, we then conclude that the coefficients of $u$ must be all identically 0, which leads to a contradiction.

Since $$Q^t_q(u, u) \geq \min\{t, 1\}(Q_q(u, u) + \|\nabla u\|^2),$$ we have $$\lambda_1^{t,q}(\Omega) \geq C \min\{t, 1\},$$ where $C > 0$ is the smallest eigenvalue of $\Box^t_q$ with $t_0 = 1$. Inequality (2.8) is then a consequence of the above inequality. \hfill $\square$

**Remark 1.** When $\Omega$ is pseudoconvex, it follows from Hörmander’s $L^2$-estimates for the $\overline{\partial}$-operator that

(2.10) $$Q^t_q(u, u) \geq Q_q(u, u) \geq \frac{q}{D^2 e} \|u\|^2, \quad u \in \text{Dom}(Q^t_q)$$

and

(2.11) $$\|N^t_q u\| \leq \frac{D^2 e}{q} \|u\|,$$

where $D$ is the diameter of $\Omega$ ([H65]; see also [CS99, Theorem 4.4.1]).

Let $S_i$, $i = 1, 2$, be non-negative self-adjoint operators on Hilbert space $\mathbb{H}$ with associated quadratic forms $Q_i$. One way to estimate the difference of variational eigenvalues of $S_1$ and $S_2$ is to construct a transition operator $T : \text{Dom}(Q_1) \to \text{Dom}(Q_2)$ and estimate the difference between $\langle f, g \rangle_1$ and $\langle Tf, Tg \rangle_2$ and that between $Q_1(f, g)$ and $Q_2(Tf, Tg)$ for any $f, g \in \text{Dom}(Q_1)$. The following simple well-known lemma is useful (see, e.g., [FZ19, Lemma 2.1]).

**Lemma 2.2.** Let $k$ be a positive integer. Suppose there exist $0 < \alpha_k < 1/(2k)$ and $\beta_k > 0$ such that for any orthonormal set $\{u_1, u_2, \ldots, u_k\} \subset \text{Dom}(Q_1)$,

(2.12) $$|\langle Tu_h, Tu_l \rangle_2 - \delta_{hl}| \leq \alpha_k \quad \text{and} \quad |Q_2(Tu_h, Tu_l) - Q_1(u_h, u_l)| \leq \beta_k.$$  

Then

(2.13) $$\lambda_k(S_2) \leq \lambda_k(S_1) + 2k(\alpha_k \lambda_k(S_1) + \beta_k).$$

**Remark 2.** Condition (2.12) in Lemma 2.2 can be replaced by the following: For any $k$-dimensional subspace $L_k$ of $\text{Dom}(Q_1)$ and $u \in L_k$,

(2.14) $$\|Tu\|^2 \geq (1 - k\alpha_k)\|u\|^2_1 \quad \text{and} \quad Q_2(Tu, Tu) \leq Q_1(u, u) + k\beta_k\|u\|^2_1.$$  

We refer the reader to [FZ19] for a proof of Lemma 2.2.
Spectral stability can also be studied from the perspective of resolvent convergence. Let $T_j$ and $T$ be self-adjoint operators on Hilbert space $\mathbb{H}$. Recall that $T_j$ is said to converge to $T$ in norm resolvent sense if for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the resolvent operator $R_\lambda(T_j) = (T_j - \lambda I)^{-1}$ converges to $R_\lambda(T) = (T - \lambda I)^{-1}$ in norm, and $T_j$ is said to converge to $T$ in strong resolvent sense if $R_\lambda(T_j)$ converges strongly to $R_\lambda(T)$. It is well known that if $T_j$ converges to $T$ in norm resolvent sense, then for any $\lambda \notin \sigma(T)$, $\lambda \notin \sigma(T_j)$ for sufficiently large $j$, and if $T_j$ converges to $T$ in strong resolvent sense, then for any $\lambda \in \sigma(T)$, there exist $\lambda_j \in \sigma(T_j)$ such that $\lambda_j \to \lambda$. We refer the reader to [RS80, §VIII.7] for relevant material.

### 3. Spectral stability under the elliptic regularization

In this section, we study spectral stability of the Kohn-Nirenberg Laplacian $\square^\ell_q$ as $t \to 0^+$. We obtain quantitative estimates for the difference between $\lambda^k_+(\Omega)$ and $\lambda^l_q(\Omega)$ when $\Omega$ is a smooth bounded pseudoconvex domain of finite type in the sense of D’Angelo. We also study the convergence of $\square^\ell_q$ in resolvent sense as $t \to 0^+$.

A notion of finite type was introduced by D’Angelo [Dan82] in connection with subelliptic theory of the $\bar{\partial}$-Neumann Laplacian. Roughly speaking, the $D_q$-type of a smooth bounded domain $\Omega$ is the maximal order of contact of $\partial \Omega$ with any $q$-dimensional complex analytic variety. (We refer the readers to [Dan82, Dan93, DK99] for the precise definition.) Catlin [Ca83, Ca87] showed that a smooth bounded pseudoconvex domain $\Omega$ is of finite $D_q$-type if and only if there exist constants $0 < \alpha \leq 1/2$ and $C > 0$ such that the following subelliptic estimate holds:

\[
\|u\|_\alpha^2 \leq CQ\|u, u\|, \quad u \in \text{Dom}(Q).
\]

The constant $\alpha$ is usually referred to as the order of subellipticity for the $\bar{\partial}$-Neumann Laplacian and it is equal to $1/2$ when $\Omega$ is strongly pseudoconvex. The following lemma is a direct consequence of Catlin’s theorem.

**Lemma 3.1.** Let $\Omega$ be a smooth bounded pseudoconvex domain of finite $D_q$-type in $\mathbb{C}^n$. Let $m$ be an eigenform of the $\bar{\partial}$-Neumann Laplacian $\square^\ell_q$ with associated eigenvalue $\lambda(\Omega)$. Let $m$ and $l$ be non-negative integers. Then there exist positive constants $\alpha \in (0, 1/2]$, $B_m$ and $C_l$ such that

\[
\|u\|_{W^{2\alpha}} \leq B_m(\lambda(\Omega))^m\|u\|
\]

and

\[
\|u\|_{C_l(\Omega)} \leq C_l(\lambda(\Omega))^{[\frac{n+l}{2\alpha}]+1}\|u\|.
\]

(Hereafter we use $[a]$ to denote the integer part of a real number $a$.)

**Proof.** From above-mentioned work of Catlin, we know that there exist constants $\alpha \in (0, 1/2]$ and $C_s > 0$ such that

\[
\|N_qu\|_{s+2\alpha} \leq C_s\|u\|_s.
\]

Starting with $s = 0$ and repeatedly applying (3.4) to $\square^\ell_q u = \lambda(\Omega)u$, we obtain (3.2). The estimate (3.3) is then an immediate consequence of the Sobolev embedding theorem. \(\square\)

Recall that $C^1(\Omega) \cap \text{Dom}(\bar{\partial}^*_{q-1})$ is dense in $\text{Dom}(Q_q)$ in the graph norm

\[
\|u\|_Q = (\|u\|^2_\Omega + Q\|u, u\|)^{1/2}
\]
when \( \partial \Omega \) is \( C^2 \)-smooth (see, e.g., [CS99, Lemma 4.3.2]). Thus \( \text{Dom}(Q_{t}^q) \) is also dense in \( \text{Dom}(Q_{t}) \) in the graph norm. We will use this fact in proving the following theorem:

**Theorem 3.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary. Let \( 1 \leq q \leq n - 1 \) and \( k \in \mathbb{N} \). Then

\[
\lim_{t \to 0^+} \lambda_{k,q}^t(\Omega) = \lambda_k^q(\Omega).
\]

Furthermore, if \( \Omega \) is a smooth bounded pseudoconvex domain of finite \( D_q \)-type, then there exist positive constants \( \alpha \in (0, 1/2] \) and \( C \) independent of \( t \) or \( k \) such that

\[
|\lambda_{k,q}^t(\Omega) - \lambda_k^q(\Omega)| \leq C t k (\lambda_k^q(\Omega))^{2(\lfloor \frac{n+1}{2} \rfloor + 1)}.
\]

**Proof.** On the one hand, from Proposition 2.1 we know that \( \lambda_k^q(\Omega) \leq \lambda_{k,q}^t(\Omega) \). On the other hand, since \( W_{(0,q)}^1(\Omega) \cap \text{Dom}(\mathcal{D}_{q-1}) \) is dense in \( \text{Dom}(Q_q) \) in the graph norm \( \| \cdot \|_q \), for any \( \varepsilon > 0 \), there exists a \( k \)-dimensional subspace \( L_k \subset W_{(0,q)}^1(\Omega) \cap \text{Dom}(\mathcal{D}_{q-1}) \) such that

\[
\lambda_k^q(\Omega) + \varepsilon \geq \lambda_Q(L_k) = \sup_{u \in L_k \setminus \{0\}} \frac{Q^t(u,u) - t \|\nabla u\|^2}{\|u\|^2}.
\]

(3.7)

\[
\geq \sup_{u \in L_k \setminus \{0\}} \frac{Q^t(u,u)}{\|u\|^2} - t \sup_{u \in L_k \setminus \{0\}} \frac{\|\nabla u\|^2}{\|u\|^2} \geq \lambda_{k,q}^t(\Omega) - t \sup_{u \in L_k \setminus \{0\}} \frac{\|\nabla u\|^2}{\|u\|^2}.
\]

Letting \( t \to 0^+ \), we then have \( \limsup_{t \to 0} \lambda_{k,q}^t(\Omega) \leq \lambda_k^q(\Omega) \) and hence (3.5).

Under the pseudoconvexity and finite type assumptions, the spectrum of \( \Box_q \) is purely discrete. Let \( u_l \) be eigenforms associated with eigenvalues \( \lambda_l^q(\Omega) \), \( 1 \leq l \leq k \). Let \( L_k = \text{Span}\{u_1, u_2, \ldots, u_k\} \) and let \( u \in L_k \). It follows from Lemma 3.1 and the Cauchy-Schwarz inequality that there exist constants \( \alpha \in (0, 1/2] \) and \( C > 0 \) such that

\[
\|\nabla u\|^2 \leq C k (\lambda_k^q(\Omega))^{2(\lfloor \frac{n+1}{2} \rfloor + 1)} \|u\|^2.
\]

Thus

\[
0 \leq \lambda_{k,q}^t(\Omega) - \lambda_k^q(\Omega) \leq t \sup_{u \in L_k \setminus \{0\}} \frac{\|\nabla u\|^2}{\|u\|^2} \leq C t k (\lambda_k^q(\Omega))^{2(\lfloor \frac{n+1}{2} \rfloor + 1)}.
\]

This concludes the proof of Theorem 3.2. \( \square \)

We now study the resolvent convergence of the Kohn-Nirenberg Laplacian \( \Box_q^t \) as \( t \to 0^+ \). Our result is as follows.

**Theorem 3.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary. Then \( \Box_q^t \) converges to \( \Box_q \) in strong resolvent sense as \( t \to 0^+ \). If \( \Omega \) is strongly pseudoconvex with smooth boundary, then \( \Box_q^t \) converges to \( \Box_q \) in norm resolvent sense.

**Proof.** Let \( \tilde{Q}_q(u,v) = Q_q(u,v) + \langle u, v \rangle \) and let

\[
F_q = \Box_q + I \quad \text{and} \quad R_q = (\Box_q + I)^{-1}.
\]

Similarly, let \( \tilde{Q}_q^t(u,v) = Q_q^t(u,v) + \langle u, v \rangle \) and let

\[
F_q^t = \Box_q^t + I \quad \text{and} \quad R_q^t = (\Box_q^t + I)^{-1}.
\]
For any $u \in L^2_{(0,q)}(\Omega)$, $R_q(u) \in \text{Dom}(\square_q) \subset \text{Dom}(Q_q)$. Since $C^1_{(0,q)}(\overline{\Omega}) \cap \text{Dom}(\overline{\partial}_{q-1})$ is dense in $\text{Dom}(Q_q)$ in the graph norm $\| \cdot \|_Q$, for any $\epsilon > 0$, there there exists $v \in W^1_{(0,q)}(\Omega) \cap \text{Dom}(\overline{\partial}_{q-1})$ such that

$$\|v - R_q u\|_Q < \epsilon.$$  

Note that $R_q(u) \in \text{Dom}(\square_q) \subset \text{Dom}(Q_q^\epsilon) \subset \text{Dom}(Q_q)$. We have

$$\|R_q^\epsilon u - R_q u\|^2 \leq \tilde{Q}_q(R_q^\epsilon u - R_q u, R_q^\epsilon u - R_q u)$$

$$= \tilde{Q}_q(R_q^\epsilon u, R_q^\epsilon u - v) + \tilde{Q}_q(R_q^\epsilon u, v - R_q u) - \langle u, R_q^\epsilon u - R_q u \rangle$$

$$= \langle u, R_q u - v \rangle - t\langle \nabla R_q^\epsilon u, \nabla(R_q^\epsilon u - v) \rangle + \tilde{Q}_q(R_q^\epsilon u, v - R_q u)$$

$$\leq \|u\|\|R_q u - v\| + t\|\nabla R_q^\epsilon u\|\|\nabla v\| + \|R_q^\epsilon u\|_Q \|v - R_q u\|_Q.$$  

Note that

$$t\|\nabla(R_q^\epsilon u)\|^2 \leq \tilde{Q}_q(R_q^\epsilon u, R_q^\epsilon u) = \langle u, R_q^\epsilon u \rangle \leq \|u\|\|R_q^\epsilon u\| \leq \|u\|^2$$

and

$$\|R_q^\epsilon u\|_Q^2 \leq \tilde{Q}_q(R_q^\epsilon u, R_q^\epsilon u) = \langle u, R_q^\epsilon u \rangle \leq \|u\|\|R_q^\epsilon u\| \leq \|u\|^2.$$  

It follows that

$$\|R_q^\epsilon u - R_q u\|^2 \leq \|u\| \left( t^{1/2}\|\nabla v\| + 2\epsilon \right).$$

Letting $t \to 0^+$ and then $\epsilon \to 0^+$, we then conclude that $\|R_q^\epsilon u - R_q u\| \to 0$ as $t \to 0^+$. When $\Omega$ is pseudoconvex, the spectra of $\square_q$ and $\square_q$ are both contained in the interval $[q/eD^2, \infty)$, where as before $D$ is the diameter of $\Omega$ ([H65]; see also, e.g., [CS99, Theorem 4.4.1]). Thus in this case, it suffices to consider the convergence of the $\overline{\partial}$-Green operator $N_q^\epsilon$ (see, e.g., [RS80, Theorem VIII.9]). When $\Omega$ is strongly pseudoconvex with smooth boundary, from Kohn’s subelliptic estimate we know that there exists a constant $C > 0$ such that

$$\|N_q u\|_1 \leq C\|u\| \quad \text{and} \quad \|N_q^\epsilon u\|_1 \leq C\|u\|$$

for any $u \in L^2_{(0,q)}(\Omega)$ ([Ko63, FK72]). Thus $N_q u \in \text{Dom}(Q_q^\epsilon)$ and we have

$$Q_q(N_q^\epsilon u, N_q^\epsilon u - N_q u) = Q_q(N_q^\epsilon u, N_q^\epsilon u - N_q u) - t\langle \nabla N_q^\epsilon u, \nabla(N_q^\epsilon u - N_q u) \rangle$$

$$= \langle u, N_q^\epsilon u - N_q u \rangle - t\langle \nabla N_q^\epsilon u, \nabla(N_q^\epsilon u - N_q u) \rangle$$

$$= Q_q(N_q u, N_q^\epsilon u - N_q u) - t\langle \nabla N_q^\epsilon u, \nabla(N_q^\epsilon u - N_q u) \rangle.$$  

Therefore

$$\frac{q}{eD^2}\|N_q^\epsilon u - N_q u\|^2 \leq Q_q(N_q^\epsilon u, N_q^\epsilon u - N_q u)$$

$$= Q_q(N_q^\epsilon u, N_q^\epsilon u - N_q u) - Q_q(N_q u, N_q^\epsilon u - N_q u)$$

$$= -t\langle \nabla N_q^\epsilon u, \nabla(N_q^\epsilon u - N_q u) \rangle \leq t\langle \nabla N_q^\epsilon u, \nabla N_q u \rangle$$

$$\leq t\|\nabla N_q^\epsilon u\|\|\nabla N_q u\| \leq Ct\|u\|^2.$$  

Hence $N_q^\epsilon$ converges to $N_q$ in norm as $t \to 0^+$.  

Remark 3. To establish that $\square^t_q$ converges to $\square_q$ in resolvent sense, it suffices to prove that $(\square^t_q - \lambda)^{-1}$ converges correspondingly to $(\square_q - \lambda)^{-1}$ for some $\lambda < 0$. This is a direct consequence of the second resolvent identity (see, [W80, Theorem 5.13]):

$$(\square^t_q - \lambda)^{-1} - (\square_q - \lambda)^{-1} = (\square^t_q - \lambda)^{-1}(\square_q + i) [(\square_q + i)^{-1} - (\square_q - \lambda)^{-1}] (\square_q + i)(\square_q - \lambda)^{-1}.$$

Remark 4. One cannot expect that $\square^t_q$ converges to $\square_q$ in norm resolvent sense if $\Omega$ is only assumed to be weakly pseudoconvex with smooth boundary. For example, if $\partial \Omega$ contains an $(n - 1)$-dimensional complex analytic variety, then by Proposition 2.1, $N_q^t$ is compact but $N_q$ is not (see [FS01]). Hence $N_q^t$ cannot converge to $N_q$ in norm.

4. Spectral stability under domain perturbation

Our aim in this section is to establish a quantitative estimate for $|\lambda^t_q(\Omega_1) - \lambda^t_q(\Omega_2)|$ when $\Omega_1$ and $\Omega_2$ are smooth bounded domains in $\mathbb{C}^n$ that are sufficiently close to each other. The key is to construct a transition operator $T$ form $\text{Dom}(Q^t_q,\Omega_1)$ to $\text{Dom}(Q^t_q,\Omega_2)$ such that $||Tu||_{\Omega_2} = ||u||_{\Omega_1}$ and $|Q^t_q,\Omega_1(Tu,Tu) - Q^t_q,\Omega_2(u,u)|$ is controlled by the closeness between $\Omega_1$ and $\Omega_2$. (Here we use $Q^t_q,\Omega$ to denote the quadratic form associated with $\square^t_q$ acting on $(0,q)$-forms on $\Omega$. To economize the notation, we will sometimes drop the subscript $q$ when doing so causes no confusion.) Since $\text{Dom}(Q^t_q,\Omega_1) = W^{1,1}_{(0,q)}(\Omega) \cap \text{Dom}(\overline{\square})$, the restriction of a form from $\text{Dom}(Q^t_q,\Omega_1)$ no longer belongs to $\text{Dom}(Q^t,V)$ where $V$ is a subdomain of $\Omega$. Additionally, the extension of a form from $\text{Dom}(Q^t_q,\Omega_1)$ to zero outside of $\Omega$ does not make it belong to $\text{Dom}(Q^t_q,\Omega_2)$ where $V$ is a larger domain containing $\Omega$. As in [FZ19], we overcome these difficulties by decomposing $u \in \text{Dom}(Q^t_q,\Omega_1)$ into the tangential and normal components and treat them separately. The tangential component is dealt with as in the case of the Neumann Laplacian while the normal component is handled as in the case of the Dirichlet Laplacian.

We now elaborate on how to measure the closeness between domains. Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ with $C^m$-smooth boundary $(2 \leq m \leq \infty)$. A real valued function $r \in C^m(\mathbb{C}^n)$ is said to be a defining function of $\Omega$ if $r < 0$ on $\Omega$, $r > 0$ on $\mathbb{C}^n \setminus \overline{\Omega}$, and $|\nabla r| \not\equiv 0$ on $\partial \Omega$. The defining function is normalized if $|\nabla r| = 1$ on $\partial \Omega$. Let $\rho$ be the signed distance function of $\Omega$ such that $\rho(z) = -\text{dist}(z,\partial \Omega)$ when $z \in \Omega$ and $\rho(z) = \text{dist}(z,\partial \Omega)$ when $z \in \mathbb{C}^n \setminus \overline{\Omega}$. It is well known that there is a neighborhood $U$ of $\partial \Omega$ such that $\rho \in C^m(U)$ (see [KP81]). It follows that for any normalized defining function $r(z)$ of $\Omega$, we have $r(z) = h(z)\rho(z)$ for some positive function $h \in C^{m-1}(U)$ such that $h = 1$ on $\partial \Omega$. For $\delta > 0$, let

$$\Omega_\delta^- = \{ z \in \mathbb{C}^n \mid r(z) < -\delta \} \quad \text{and} \quad \Omega_\delta^+ = \{ z \in \mathbb{C}^n \mid r(z) < \delta \}.$$

Let $\Omega_j$ be a bounded domain in $\mathbb{C}^n$ with $C^m$-smooth boundary. Let $r_j$ be a normalized defining function for $\Omega_j$. The closeness between $\Omega$ and $\Omega_j$ will be measured by $\delta_j = ||r - r_j||_{C^2}$, the $C^2$-norm of $r - r_j$ over $\overline{\Omega} \cup \overline{\Omega}_j$. Note that for any $a > 1$,

$$\Omega_{\delta_j} \mathrel{\subset} \Omega_j \subset \Omega_{\delta_j}^a$$

provided $\delta_j$ is sufficiently small. Furthermore, the signed distance function $\rho_j$ of $\Omega_j$ is $C^m$ on some neighborhood $U$ of $\partial \Omega$ (see [Fe59, Lemma 4.11] and [KP81, Theorem 3]).
Lemma 4.1. Let $\Omega$ be a smooth bounded domain in $\mathbb{C}^n$. Let $s$ be a non-negative integer. Then there exists a constant $C > 0$ independent of $t$ such that

\begin{equation}
\|u\|_{W^{s+2}} \leq \frac{C}{t^{2s+1}} \|\Box_t u\|_{W^s},
\end{equation}

for all $u \in \text{Dom}(\Box_t)$ with $\Box_t u \in W^{s}_{(0,q)}(\Omega)$. Moreover, if $\Omega$ is pseudoconvex, then the above estimate can be improved as follows:

\begin{equation}
\|u\|_{W^{s+2}} \leq \frac{C}{t^{2s+1}} \|\Box_t u\|_{W^s}.
\end{equation}

Proof. The proof follows the same line of arguments as in the proof of Proposition 3.5 in [S10]. One just needs to keep track of the constants. We provide the details for the proof of (4.1). By Proposition 2.1, we have

\begin{equation}
Ct \|u\|^2 \leq Q^t(u,u), \quad u \in \text{Dom}(Q^t),
\end{equation}

where the constant $C > 0$ is independent of $t$. Hence

\begin{equation}
\|u\|_{W^1(\Omega)} \leq \frac{C}{\sqrt{t}} (Q^t(u,u))^{1/2} \leq \frac{C}{\sqrt{t}} \|\Box_t u\|_{W^s}^{1/2} \|u\|_{W^s}^{1/2}, \quad u \in \text{Dom}(\Box^t).
\end{equation}

It follows from (2.8) that $\|u\| \leq (C/t) \|\Box_t u\|$. Therefore

\begin{equation}
\|u\|_{W^1} \leq \frac{C}{t} \|\Box_t u\|.
\end{equation}

It suffices to prove (4.1) when $u$ is supported in a special boundary chart. The general case is obtained by a partition of unity argument. Let $(t_1, \ldots, t_{2n-1}, r)$ be a local special coordinate chart near a boundary point where $r$ is a defining function of $\Omega$ and $(t_1, \cdots, t_{2n-1})$ are coordinates on the boundary. Denote by $D_j^h$, $1 \leq j \leq 2n - 1$, the difference quotient with respect to $t_j$, acting on forms coefficientwise in a special boundary frame associated to special boundary chart. Note that $D_j^h$ preserves $\text{Dom}(\overline{D}^*)$. For any $(0,q)$-form $\nu \in \text{Dom}(Q^t)$, we have

\begin{align*}
|\langle \Box^t D_j^h u, v \rangle| &= |\langle \overline{D}^t D_j^h u, \overline{D}^* v \rangle + \langle \overline{D}^t D_j^h u, \overline{D}^* v \rangle + t\langle \nabla D_j^h u, \nabla v \rangle | \\
&\leq C \|u\|_{W^1} \|v\|_{W^1} + |\langle D_j^h \overline{D} u, \overline{D}^* v \rangle + \langle D_j^h \overline{D} u, \overline{D}^* v \rangle + t\langle D_j^h \nabla u, \nabla v \rangle | \\
&= C \|u\|_{W^1} \|v\|_{W^1} + |\langle \overline{D} u, D_j^{-h} \overline{D}^* v \rangle + \langle \overline{D} u, D_j^{-h} \overline{D}^* v \rangle + t\langle \nabla u, D_j^{-h} \nabla v \rangle | \\
&\leq C \|u\|_{W^1} \|v\|_{W^1} + |Q^t(u, D_j^{-h} v) | \leq \frac{C}{t} \|\Box^t u\| \|v\|_{W^1}.
\end{align*}

Substituting $v$ by $D_j^h u$ in the above estimate, we obtain

\begin{equation}
\|D_j^h u\|^2_{W^1} \leq \frac{C}{t} Q^t(D_j^h u, D_j^h u) = \frac{C}{t} \langle \Box^t D_j^h u, \Box^t D_j^h u \rangle \leq \frac{C}{t^2} \|\Box^t u\|^2 \|D_j^h u\|_{W^1}.
\end{equation}

Therefore

\begin{equation}
\|\nabla_T u\|_{W^1} \leq C \|D_j^h u\|_{W^1} \leq \frac{C}{t^2} \|\Box^t u\|,
\end{equation}

where $\nabla_T$ denotes the gradient with respect to the tangential coordinates $(t_1, \cdots, t_{2n-1})$. 

We first establish some auxiliary estimates. It follows from the elliptic theory that $\|u\|_{s+2} \leq C_{t} \|\Box^t u\|_{s}$ on a smooth bounded pseudoconvex domain (see [S10, Proposition 3.5]). The following lemma is a quantitative version of this result. Throughout this section, we will assume that $0 < t < 1$, $1 \leq q \leq n - 1$ and $n$ is a positive integer.
We now estimate the full Sobolev norm. Note that
\[ \square^t u = -(\frac{1}{4} + t) \Delta u \]
when \( u \in \text{Dom}(\square^t) \). Writing \( \Delta \) in terms of tangential and normal derivatives in the local coordinates, we obtain
\[ \left\| \frac{\partial^2 u}{\partial \nu^2} \right\| \leq C \left( \| \square^t u \| + \| u \|_{W^1} + \| \nabla_T u \|_{W^1} \right), \]
where \( \frac{\partial}{\partial \nu} \) is the normal derivative. Consequently,
\[ (4.7) \quad \| u \|_{W^2} \leq C \left( \| \nabla_T u \|_{W^1} + \| \partial^2 u/\partial \nu^2 \| \right) \leq \frac{C}{t} \| \square^t u \|. \]
Thus \( (4.1) \) holds for \( s = 0 \). We proceed with the inductive step. Assume that \( (4.1) \) holds for \( s \). Then
\[ (4.8) \quad \| D_j^h u \|_{W^{s+2}} \leq \frac{C}{t^{2s+1}} \| \square^t D_j^h u \|_{W^s} \]
\[ \leq \frac{C}{t^{2s+2}} \left( \| u \|_{s+2} + \| D_j^h \square^t u \|_{W^s} \right) \]
\[ \leq \frac{C}{t^{2s+2}} \| \square^t u \|_{W^{s+1}}. \]
Thus
\[ \| \nabla_T u \|_{W^{s+2}} \leq \frac{C}{t^{2s+2}} \| \square^t u \|_{W^{s+1}}. \]
By the same arguments proceeding \( (4.7) \), we then establish \( (4.1) \) for \( s + 1 \).

When \( \Omega \) is pseudoconvex, from Hörmander’s \( L^2 \)-estimate, we have \( \| u \| \leq C \| \square^t u \| \). Using this instead of \( (2.8) \), we obtain
\[ (4.9) \quad \| u \|_{W^1} \leq \frac{C}{\sqrt{t}} \| \square^t u \|. \]
Note that in this case, the constants in \( (4.9) \) and \( (4.3) \) depend only on the diameter of \( \Omega \) and \( q \). The rest of the proof follows from the same lines except with different exponents of \( t \). This concludes the proof of Lemma 4.1. \( \square \)

**Lemma 4.2.** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{C}^n \). Let \( u \) be an eigenform of \( \square^t \) with associated eigenvalue \( \lambda^t(\Omega) \). Let \( l \) be a non-negative integer. Then there exists a constant \( C > 0 \) independent of \( t \) such that
\[ (4.10) \quad \| u \|_{C^l(\bar{\Omega})} \leq \frac{C}{t^{2(2^{n+l+2}-1)/3}} (\lambda^t(\Omega))^{\frac{[n+l]}{2}+1+1} \| u \|, \]
where \( \lfloor (n+l)/2 \rfloor \) as before denotes the integer part of \( (n+l)/2 \). Furthermore, if \( \Omega \) is pseudoconvex, then
\[ (4.11) \quad \| u \|_{C^l(\bar{\Omega})} \leq \frac{C}{t^{(2^{n+l+2}-1)/2}} (\lambda^t(\Omega))^{\frac{[n+l]}{2}+1} \| u \|. \]
Proof. This is a direct consequence of Lemma 4.1 and the Sobolev embedding theorem. We provide only the proof for (4.11).

For \( u \in \text{Dom}(\Box^s) \), from (4.1) with \( s = 0 \), we have

\[
\|u\|_{W^2} \leq C \frac{1}{t^{3/2}} \|\Box u\| \leq C \frac{1}{t^{3/2}} \lambda^t(\Omega)\|u\|.
\]

Thus \( u \in W^2_{(0,q)}(\Omega) \) and \( \Box u = \lambda^t(\Omega)u \in W^2_{(0,q)}(\Omega) \). From (4.1) with \( s = 2 \), we obtain

\[
\|u\|_{W^4} \leq C \frac{1}{t^6} \|\Box u\|_{W^2} \leq C \frac{1}{t^{15/2}} (\lambda^t(\Omega))^2\|u\|.
\]

Repeating this process, we obtain \( u \in W^2_{(0,q)}(\Omega) \) and

\[
\|u\|_{W^{2m}(\Omega)} \leq C \frac{1}{t^{(2m-1)/2}} (\lambda^t(\Omega))^m\|u\|, \quad m \in \mathbb{N}.
\]

The desired inequality (4.11) is then an immediate consequence of Sobolev embedding theorem. \( \square \)

Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{C}^n \) with \( C^m \)-smooth boundary (\( m \geq 2 \)). Let \( r(z) \) be a normalized defining function of \( \Omega \). Then \( |\nabla r(z)| = 1 \) on \( \partial \Omega \). Let \( z' \in \partial \Omega \) and let \( U' \subset U \) be a tubular neighborhood of \( z' \) such that \( |\nabla r(z) - \nabla r(z')| < 1/2 \) when \( z \in U' \) and

\[
\Omega^\pm = \{ z \in \mathbb{C}^n \mid r(z) < \pm \delta \}.
\]

Shrinking \( U' \) if necessary, then for sufficiently small \( \delta > 0 \), we have \( z - 2\delta \vec{n}(z') \in \Omega \) for all \( z \in U' \cap \Omega^+_\delta \) and \( z + 2\delta \vec{n}(z') \notin \Omega \) for all \( z \in U' \cap \Omega^-_\delta \). Furthermore,

\[
\text{dist}(z - 2\delta \vec{n}(z'), \partial \Omega) \geq \text{dist}(z - 2\delta \vec{n}(z), \partial \Omega) - 2\delta |\vec{n}(z) - \vec{n}(z')| > 2\delta - \delta = \delta,
\]

for all \( z \in U' \cap \Omega^+_\delta \). We choose a finite covering \( \{V^l\}_{l=0}^m \) of \( \Omega \) such that \( V^0 \subset \subset \Omega \) and each \( V^l, 1 \leq l \leq m, \) is a tubular neighborhood about some \( z^l \in \partial \Omega \) constructed as above. Write \( \vec{n}^l = \vec{n}(z^l) \). We then have

\[
\bigcup_{l=1}^m \left\{ z - 2\delta \vec{n}^l \mid z \in V^l \cap \Omega \right\} \subset V^0 \subset \Omega^-_\delta
\]

and

\[
\bigcup_{l=1}^m \left\{ z + 2\delta \vec{n}^l \mid z \in V^l \cap \Omega \right\} \subset V^0 \supset \Omega^+_\delta.
\]

We now construct a frame for \( (0,1) \)-forms on \( V^l \). Since \( |\nabla r| > 1/2 \) on \( V^l \). Shrinking \( V^l \) if necessary, we may assume without loss of generality that \( \partial r/\partial z_1 \neq 0 \) on \( V^l \). Let

\[
\vec{\omega}^k_l = \partial r \quad \text{and} \quad \vec{\omega}^k_l = d\bar{z}_k - \left(4\partial r/\partial z_k\right)\partial r, \quad 1 \leq k \leq n - 1.
\]

Note that since \( |\nabla r| = 1 \) on \( \partial \Omega \). Hence \( \vec{\omega}^k_l, 1 \leq k \leq n - 1 \), is pointwise orthogonal to \( \vec{\omega}^k_l \) on \( \partial \Omega \) and satisfies the \( \partial \)-Neumann boundary condition on \( V^l \cap \partial \Omega \). Furthermore, the determinant of the coefficients of the \( (0,1) \)-forms \( \vec{\omega}^k_l, 1 \leq k \leq n \), is non-zero on \( V^l \). Thus \( \{\vec{\omega}_1^k, \ldots, \vec{\omega}_n^k\} \) is indeed a frame for the \( (0,1) \)-forms on \( V^l \). Let \( \Omega_j \) be a bounded domain with \( C^m \)-boundary with a normalized defining function \( r_j \) such that \( \|r_j - r\|_{C^2} \) is sufficiently small. We then construct a frame \( \{\vec{\omega}_{j,l}^k; 1 \leq k \leq n\} \) as above but with \( r \) replaced by \( r_j \). Thus we have

\[
(4.12) \quad \|\vec{\omega}^k_l - \vec{\omega}_{j,l}^k\|_{C^1(V^l \cap (\Omega_j \cup \Omega))} \leq C\|r - r_j\|_{C^2(\Omega_j \cup \Omega)}.
\]
Let \( \{\psi^j\}_{j=1}^m \) be a partition of unity subordinated the covering \( \{V^l, 0 \leq l \leq m\} \) such that \( \text{supp } \psi^j \subset V^l \). Let \( \mathcal{E} : W^s(\Omega) \to W^s(\mathbb{C}^n) \) be a continuous extension operator. Recall that the norm of this operator depends only on \( n, s \), and the Lipschitz constant of \( \Omega \) ([St70, Ch VI.3, Theorem 5]). Let \( d(z) = \text{dist}(z, \partial \Omega) \). Let \( \chi_j(t) \) be a smooth function such that \( \chi_j(t) = 0 \) if \( t > 2\delta_j \), \( \chi_j(t) = 2\delta_j \) if \( t < \delta_j \), and \( 0 \leq \chi_j'(t) \leq 2 \).

We are now in position to define the transition operator. Let \( u^t \in C^\infty_{(0,q)}(\overline{\Omega}) \cap \text{Dom}(Q^l_\Omega) \). Using the partition of unity, we write

\[
(4.13) \quad u^t = \psi^0 u^t + \sum_{l=1}^m \sum_{|j|=q} \psi^j u^t_j \overline{\omega}^j,
\]

where \( \{\overline{\omega}^1_j, \cdots, \overline{\omega}^m_j\} \) is the frame for \( (0,1) \)-forms on \( V^l \) \( (1 \leq l \leq m) \) constructed as above. Since \( u^t \in \text{Dom}(Q^l_\Omega) \), we have \( u^t_j = 0 \) on \( \partial \Omega \) when \( n \in J \). We extend such \( u^t_j \)'s to be zero outside of \( \Omega \). Define \( T_j : C^\infty_{(0,q)}(\overline{\Omega}) \cap \text{Dom}(Q^l_\Omega) \to C^\infty_{(0,q)}(\overline{\Omega}) \cap \text{Dom}(Q^l_\Omega) \) by

\[
(4.14) \quad T_j u^t = \psi^0 u^t + \sum_{l=1}^m \left( \sum_{J \cap \overline{n} \notin J} \mathcal{E}[\psi^j u^t_j] \overline{\omega}^j + \sum_{J \cap \overline{n} \in J} \psi^j(z) u^t_j(z + \chi(d(z)) \overline{n}) \overline{\omega}^j \right),
\]

where \( \{\overline{\omega}^1_j, \cdots, \overline{\omega}^m_j\} \) is the local frame of \( (0,1) \)-forms constructed as above on \( V^l \). Notice that \( T_j u^t \in \text{Dom}(Q^l_\Omega) \) because the coefficients of \( \overline{\omega}^j \) in the above expression is 0 near \( \partial \Omega \) if \( n \in J \).

**Theorem 4.3.** Let \( \Omega \) and \( \Omega_j \) be smooth bounded domains in \( \mathbb{C}^n \) with normalized defining functions \( r \) and \( r_j \) respectively. Let \( 1 \leq q \leq n - 1 \) and \( k \in \mathbb{N} \). Then there exist constants \( \delta \) and \( C_k > 0 \) independent of \( t \) and \( j \) such that

\[
(4.15) \quad \lambda_{k}^{t,q}(\Omega_j) \leq \lambda_{k}^{t,q}(\Omega) + \frac{C_k \delta_j}{\epsilon(2^{n+3}-1)^3},
\]

provided \( \delta_j = \|r - r_j\|_{C^2} < \delta \). Furthermore, if \( \Omega \) is pseudoconvex, then

\[
(4.16) \quad \lambda_{k}^{t,q}(\Omega_j) \leq \lambda_{k}^{t,q}(\Omega) + \frac{C_k \delta_j}{\epsilon(2^{n+3}-1)}. \]

**Proof.** We provide only the proof for (4.16). The proof of (4.15) follows exactly the same lines. Let \( L_k = \{u^t_1, \cdots, u^t_k\} \), where \( \square_{\Omega} u^t_h = \lambda_{k}^{t,q}(\Omega) u^t_h \) and \( \langle u^t_h, u^t_l \rangle_{\Omega} = \delta_{hl} \) for \( 1 \leq h, l \leq k \). From elliptic theory (see Lemma 4.2 above), we know that all the eigenforms \( u^t_i \) are smooth up to the boundary. We first estimate \( \left| \langle T_j u^t_h, T_j u^t_l \rangle_{\Omega_j} - \langle u^t_h, u^t_l \rangle_{\Omega} \right| \). Note that \( \Omega_{2\delta_j} \subset \Omega \cap \Omega_j \). We have

\[
(4.17) \quad \left| \langle T_j u^t_h, T_j u^t_l \rangle_{\Omega_j} - \langle u^t_h, u^t_l \rangle_{\Omega} \right| \leq \left| \langle T_j u^t_h, T_j u^t_l \rangle_{\Omega_{2\delta_j}} - \langle u^t_h, u^t_l \rangle_{\Omega_{2\delta_j}} \right| + \left| \langle u^t_h, u^t_l \rangle_{\Omega_j \setminus \Omega_{2\delta_j}} \right|
\]

\[
\leq \left| \langle T_j u^t_h - u^t_h, T_j u^t_l \rangle_{\Omega_{2\delta_j}} \right| + \left| \langle u^t_h, T_j u^t_l - u^t_l \rangle_{\Omega_{2\delta_j}} \right| + \left| \langle u^t_h, u^t_l \rangle_{\Omega_j \setminus \Omega_{2\delta_j}} \right|.
\]
Since the corresponding coefficients of \( u_h^t \) and \( T_j u_h^t \) are the same on \( \Omega_{2\delta} \), we have

\[
(4.18) \quad \| T_j u_h^t - u_h^t \|_{\Omega_{2\delta}} \leq \sum_{l=1}^{m} \sum_{|J|=q} \| \psi_j^l u_{h,J}^t \|_{L^2} \| \omega_J^{t,l} - \omega_J^{t} \|_{C^0} \leq C\delta_j.
\]

Note that \( |\Omega_j \setminus \Omega_{2\delta_j}| \leq C\delta_j \). We have

\[
\langle T_j u_h^t, T_j u_l^t \rangle_{\Omega_j \setminus \Omega_{2\delta_j}} \leq \| T_j u_h^t \|_{C^0} \| T_j u_l^t \|_{C^0} |\Omega_j \setminus \Omega_{2\delta_j}|
\]

\[
(4.19) \quad \leq \frac{Ck\delta_j}{t^{2n+3-1}}.
\]

Here in the last inequality we have used (the proof of) Lemma 4.2 and the Sobolev embedding theorem and the fact that the extension map \( E: W^s(\Omega) \rightarrow W^s(\mathbb{R}^n) \) is bounded. The other terms in (4.17) are estimated similarly. We then obtain

\[
(4.20) \quad |\langle T_j u_h^t, T_j u_l^t \rangle_{\Omega_j} - \langle u_h^t, u_l^t \rangle_{\Omega_j}| \leq \frac{Ck\delta_j}{t^{2n+3-1}}.
\]

As a consequence, \( \{T_j u_h^t, \ldots, T_j u_k^t\} \) is linearly independent when \( \delta \) is sufficiently small.

The expression \( |Q_{\Omega}^t(T_j u_h^t, T_j u_l^t) - Q_{\Omega}^t(u_h^t, u_l^t)\rangle \) can be estimated in the same way. The main difference is that in this case, we will need to estimate the first derivatives of \( u_j^t \) and the second derivatives of \( r \) and \( r_j \). For example, we have

\[
\| \bar{\partial}(T_j u_h^t - u_h^t) \|_{\Omega_{2\delta_j}} \leq \sum_{l=1}^{m} \sum_{|J|=q} \left( \| \bar{\partial}(u_h^t, J) \|_{L^2} \| \omega_J^{t,l} - \omega_J^{t} \|_{C^0} + \| \psi_j^l u_{h,J}^t \|_{L^2} \| \bar{\partial}(\omega_J^{t,l} - \omega_J^{t}) \|_{C^0} \right)
\]

\[
(4.21) \quad \leq C \| u_h^t,J \|_{1} \| r_j - r \|_{C^2} \leq \frac{Ck\delta_j}{t^{2n+3-1}/2}.
\]

Therefore we have

\[
(4.22) \quad |Q_{\Omega}^t(T_j u_h^t, T_j u_l^t) - Q_{\Omega}^t(u_h^t, u_l^t)| \leq \frac{Ck\delta_j}{t^{2n+3-1}}.
\]

Applying Lemma 2.2, we then have

\[
\lambda_{k,l}^{t,q}(\Omega_j) \leq \lambda_{k,l}^{t,q}(\Omega) + \frac{Ck\delta_j}{t^{2n+3-1}}.
\]

We remark that in the above theorem, we need only \( \Omega \) to be \( C^{n+3} \)-smooth. To complete the proof of Theorem 1.2, it remains to establish the estimate in the opposite direction:

\[
(4.23) \quad \lambda_{k,l}^{t,q}(\Omega) \leq \lambda_{k,l}^{t,q}(\Omega_j) + \frac{Ck\delta_j}{t^{2n+3-1}}.
\]

The proof of (4.23) is similar to that of (4.16). In this case, the transition operator is from \( \text{Dom}(Q_{\Omega_j}^t) \) to \( \text{Dom}(Q_{\Omega}^t) \) and one needs to make sure that the constants in the proofs are independent of \( j \). Note that the constants in Lemma 4.1 and Lemma 4.2 depend only on the diameter of \( \Omega \) and the \( C^\infty \)-norm of a defining function of \( \Omega \). (In fact, only derivatives up to \( (n+3)^{th} \)-order of the defining function have been used in the proofs.) The assumption that the \( C^\infty \)-norm of the defining function \( r_j \) is uniformly bounded guarantees that the constants in the proofs of Lemma 4.1, Lemma 4.2, and Theorem 4.3 are indeed independent of \( j \) when the roles of \( \Omega_j \) and \( \Omega \) are reversed (see the proof of Theorem 5.6 in [FZ19] for related arguments). We leave the details to the interested reader.
References


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