Quadrilateral Mesh Generation with a Provably Good Aspect Ratio Bound[1]

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ABSTRACT
A mesh is the discretization of a geometric domain into small, simple shapes. The focus of this project is the generation of strictly convex quadrilateral (quad) meshes with provable quality guarantees through the conversion of good quality triangle meshes of planar straight line graphs. The conversion is achieved with an algorithm that uses the dual graph of the input triangulation to quadrangulate small groups of triangles at a time [1,3,4]. A specific goal of the project is proving an upper bound on the aspect ratios of all quads in the mesh. We implement two different metrics to measure the aspect ratios of the quads in the generated meshes. The first metric simply takes the ratio of the longest edge to the shortest edge for each quad. The disadvantage of this method is that it does not take into account the angle measures of the quads. The second approach, a method developed by John Robinson [2], utilizes both edge lengths and angle measures to calculate aspect ratio. We develop code to produce empirical results for both metrics of aspect ratio measurement and histogram plots showing the distribution of quad aspect ratios in a given mesh. With these experimental results, we aim to prove that, given a good quality input triangle mesh with a minimum angle bound, we can give a provably good upper bound on aspect ratio for the resulting quad mesh.

Keywords
Mesh Generation; Quadrilateral; Aspect Ratio.

1. PROBLEM AND MOTIVATION
Triangular and quadrilateral meshes have many practical applications in fields such as computer graphics, medical imaging and finite element (FE) analysis. The focus of this project is on examining quadrilateral mesh generation through the conversion of an existing triangle mesh of bounded size and with minimum angle α, with the goal of proving an upper bound on the aspect ratio of the resulting quad mesh in terms of α. Triangular meshes have been extensively researched, and several algorithms for producing provably good triangle meshes that optimize various criteria (such as angle bound, aspect ratio or size) have been previously implemented.

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The generation of good quality quadrilateral meshes is much less understood, however, and no known algorithm currently exists to generate quadrilateral meshes with a provably good aspect ratio bound. Optimizing quality criteria for quadrilateral meshes has great practical value as quad meshes have been shown to be more desirable than triangular meshes for certain FE-based applications. In particular, quad meshes with elements of small bounded aspect ratio imply elements that are close to the "ideal" quad, i.e. the square, and such shapes give better results for finite element simulations.

2. BACKGROUND AND RELATED WORK
A *mesh* is a discretization of a geometric domain into small, simple elements, such as triangles or quadrilaterals. A *planar straight-line graph* (PLSG) is any set of vertices and edges drawn in the Euclidean plane with straight-line edges intersecting only at shared endpoints. The *dual graph* of a triangulation is a graph with a node for every face of the triangulation with edges connecting two nodes only if the corresponding faces share a triangulation edge. The conversion algorithm we use to generate quadrilateral meshes takes as input a triangular mesh of a planar straight-line graph. The algorithm builds a spanning tree of the dual graph of the input triangulation, and processes the tree in a bottom up fashion by grouping together and quadrangulating small, triangulated regions at a time until the entire domain is converted to a quad mesh. This conversion may include the creation of additional points, known as *Steiner points*. After every level of the spanning tree has been processed, the mesh has been converted into a strictly convex quadrilateral mesh. In addition to the ratio of longest edge to shortest edge, we also implement a second method of aspect ratio measurement, detailed in the paper by John Robinson [2]. Robinson's method calculates the aspect ratio by using the opposite edge bisectors the quad and creating two rectangles. Each rectangle passes through one set of opposite midpoints of the quad edges and is perpendicular to that edge bisector (See Figure 1). The ratios of the width to the length of each rectangle are taken, and the larger of the two values is chosen as the final aspect ratio for the quad. The benefit of this approach over the longest quad edge to shortest quad edge method is that it accounts for angle measures in addition to edge lengths. For example, a rhombus will have a worse aspect ratio (> 1.0) than a square.

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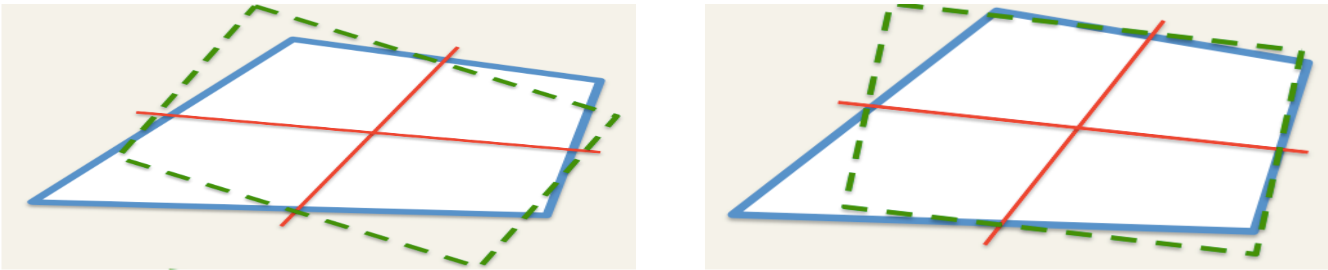

Figure 1: Rectangle constructions passing through bisectors.

3. APPROACH AND UNIQUENESS

We use previously written code implementing the triangle to quad mesh conversion algorithm [1], and we implement our own Python code to measure both metrics of aspect ratio measurement and produce histogram plots showing the aspect ratio distribution in an input quadrilateral mesh. Through the observation of these experimental results, and a more detailed study of how the conversion algorithm processes the underlying input triangular mesh, including the placement of Steiner points, we aim to prove a bound on quad aspect ratio in terms of the input triangulations minimum angle bound. To the best of our knowledge, there have been no previous attempts to prove a bound on the aspect ratio of a quadrilateral mesh through the conversion of an existing, good quality triangular mesh.

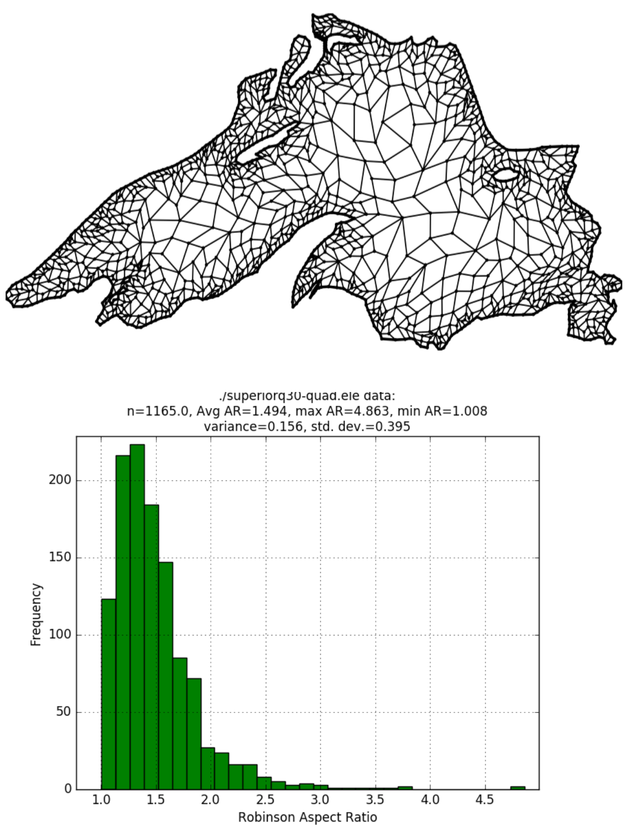

Figure 2: A quad mesh and the resulting histogram plot showing Robinson's aspect ratio distribution.

4. RESULTS AND CONTRIBUTIONS

Thus far, all results obtained have been strictly empirical. Observation of the results suggests that the aspect ratios of the quad meshes generated by the conversion method are favorable, with Robinson's aspect ratio measure generally producing better results on a given mesh than the longest edge to shortest edge measure. The quad mesh seen in figure 2 was generated by our conversion algorithm from a triangle mesh generated by the Triangle software [5] with a min. angle bound of 30° not including small angles that may be part of the input. The resulting aspect ratio histogram plot generated by our Python code for the quad mesh is shown on the right. As we can see, the aspect ratio distribution suggests a mesh of high quality. Moving forward, we aim to further study the conversion algorithm in order to better understand how the algorithm's Steiner point placement combined with the minimum angle bound of the input triangle mesh can be used to prove an aspect ratio bound for the resulting quad mesh. While we have made some preliminary progress on proving an aspect ratio bound in terms of α (min. angle bound of triangulation) for some shapes defined by small groups of triangles, we do not yet have a general result giving a provable bound for quad meshes of arbitrary PLSGs. The main challenge lies in extending these results to shapes defined by larger groups of triangles. Dr. Ramaswami and her research collaborators hope to develop a general proof for the result in the near future.

5. REFERENCES

[1] S. Ramaswami, M. Siqueira, T. Sundaram, J. Gallier, and J. Gee, "Constrained Quadrilateral Meshes of Bounded Size", *International Journal of Computational Geometry and Applications, 15(1), 2005, pp. 55-98.* (Invited paper in special issue devoted to selected papers from IMR 2003).

[2] John Robinson, "CRE Method of Element Testing and the Jacobian Shape Parameters", *Engineering Computations*, 4(2): 113-118 1987.

[3] S. Ramaswami, P. Ramos, and G. Toussaint, "Converting triangulations to quadrangulations", *Comput. Geom.: Th. Appl.* **9** (1998), pp. 257-276.

[4] D. Bremner, F. Hurtado, S. Ramaswami and V. Sacristán, "Small convex quadrangulations of point sets", *Algorithmica*, 38(2), 2003, pp. 317-339. (Invited paper in special issue devoted to selected papers from ISAAC 2001).

[5] Jonathan Shewchuk, *Triangle A Two-Dimensional Quality Mesh Generator and Delaunay Triangulator*. https://www.cs.cmu.edu/~quake/triangle.html