An optimal algorithm for Veretex Cover and Maximum Natching on Bipartite graphs

Linear independence

A collection of row vectors $\{v_i^T\}$ are *independent* if there are no constants $\{c_i\}$ so that $\sum_i c_i v_i^T = 0$.

For an $n \times n$ matrix the rows are independent if and only if the determinant is not 0.

The rank of a matrix the maximum subset of rows that are independent. The rank of the rows and the rank of the columns is the same.

This can be shown by Gaus eliminations.

The rank of a matrix

If we have Maximize $c \cdot v$ such that $Ax \geq b$, and A is an $m \times n$ type matrix. Since for every variable we have and $x \geq 0$, these rows induce the identity matrix of dimension $n \times n$. Thus the rank of the columns is n so this is also the rank of the rows.

Recall the a BFS is obtained by taking n independent *rows* and put equality and solve this $n \times n$, system of equalities. It looks as $A' \cdot x = b'$. (Note that x does not change since x had size n to begin with). Since A' has independent rows the inverse matrix ${A'}^{-1}$ exists since this is equivalent to the determinant is not 0..

Thus $x = {A'}^{-1} \cdot b$. A unique solution exists. Which is a corner (a basic feasible solution).

Summary

For minimize $c \cdot x$ under $Ax \geq b$, $x \geq 0$,

The main property we use is:

Theorem 1 *The number of independent rows is the number of variables.*

All corners or basic feasible solution are derived by taking n independent *rows and putting $A' \cdot x = b'$. The BFS is $(A')^{-1} \cdot b'$.*

The Vertex Cover problem

Given a graph $G(V, E)$, a subset $U \subseteq V$ is a *Vertex Cover* if for every edge $e = (u, v)$, either $u \in U$, or $v \in U$ or both of the above hold.

The vertex Cover problem

Input: An undirected weighted graph $G(V, E)$ with a cost function $c : V \mapsto \mathcal{Q}_+$ $c(v)$ for every v.

Required: A minimum cost subgraph U that is a Vertex Cover.

An example of a minimum Vertex Cover

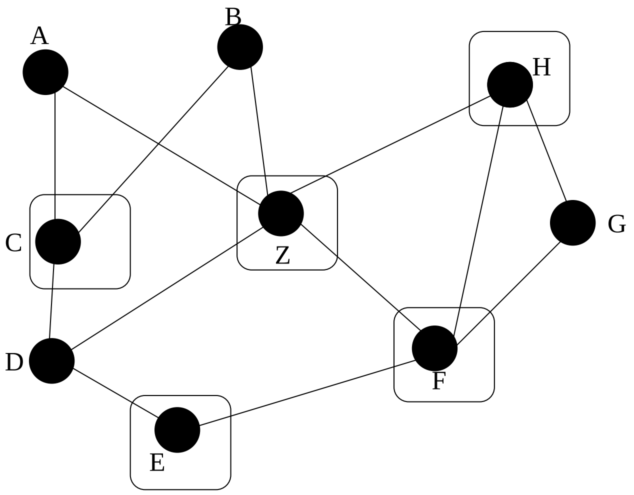


Figure 1: The squared vertices are a vertex cover. Every edge is touched by at least one squared vertex and some times EF by two of the chosen

A fractional LP for VC

$$\begin{aligned} \text{Minimize} \quad & opt_f = c_v \cdot x_v \\ & \text{subject to} \\ & x_v + x_u \geq 1 \qquad \text{for every } e = (u, v) \\ & x_v \geq 0 \end{aligned}$$

The x_v vertices in a Linear program are 1 if the vertex that is in the solution. And its zero of its not in the solution.

The main inequality $x_v + x_u \geq 1$ indicates that either v or u (or both of them, and so we can not have equality) belongs to the vertex cover.

We need to relax this to a fractional program.

Bipartite graphs

Definition 2 *A bipartite graph $G(V_1, V_2, E)$, is a graph so that $V = V_1 \cup V_2$, and there are no edges inside V_1 and no edges inside V_2, thus all edges go from V_1 to V_2.*

Such graphs are also called 2-colorable and in an equivalent definition its the collection of graphs that do not contains simple odd cycles.

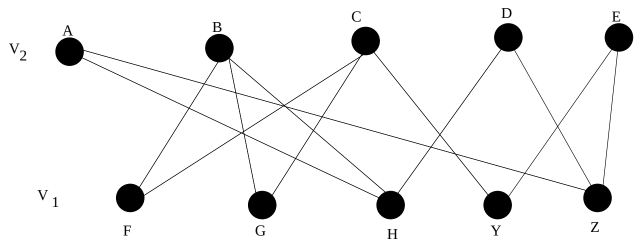
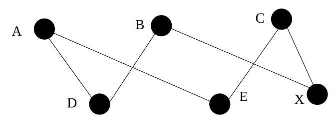


Figure 2: An example of a bipartite graph

The variables versus edges matrix of a bipartite graph



	AD	AE	BD	BX	CE	CX
A	1	1	0	0	0	0
B	0	0	1	1	0	0
C	0	0	0	0	1	1
D	1	0	1	0	0	0
E	0	1	0	0	1	0
X	0	0	0	1	0	1

Figure 3: The above LP uses the vertices versus edges matrix

The rows of the matrix of edges versus vertices

These rows are *dependent*. Note that if we add the rows of A, B, C this gives the same as adding the rows of D, EX.

In fact they both give the all 1 vector since each edge touches exactly one vertex in V_1 and exactly one vertex in V_2.

See the sum of the rows A, B, C and the sum of the rows of D, E, X.

An corollary on cycles

Corollary 3 *In a bipartite graph, the (row) vectors of the edges of a cycle are dependent.*

Proof. Any cycle is of even length and induces a subgraph $G'(V_1', V_2', E')$ with (only) the edges of the cycle. Thus Theorem is implied by the previous slide □

Thus a valid collection of rows (namely a collection of edges that induce a legal subgraph) is independent if and only if the edges induce a *forest*

An optimal iterative rounding algorithm

Theorem 4 *Say that $x_v > 0$ for every v. Then there exists a subset $F \subseteq E$ of the edges so that*

1. $x_v + x_u = 1$ *for* $e = vu$.
2. *The rows of the edges are linearly independent*
3. $|F| = |V|$

This follows from the characterization of a BFS.

An optimum algorithm

Let $\delta(v)$ be the edges touching v.

1. Set $U \leftarrow \emptyset$
2. **While** $V(G) \neq \emptyset$ **do**
   (a) Find an optimum solution for the above LP
   (b) If there is a vertex with $x_v = 0$ and $deg(_E(v) = 0$ remove this vertex
   (c) If there is a vertex with $x_v = 1$, add v to u
   and set $U \leftarrow U \cup \{v\}$ and $E \leftarrow E \setminus \delta(v)$
3. Return U

Correctness

We need to show that in any iteration there is a vertex v so that $x_v = 1$ or $x_v = 0$. Then optimality follows like in the *Assignment problem*

Claim 1 *In every iteration either we find a vertex v with $x_v = 0$ and $deg_E(v) = 0$ (for the remaining edges), or we find a vertex v with $x_v = 1$.*

Proof

For the sake of contradiction, assume that Claim 1 is false.

Thus for every vertex $x_v < 1$ and if $x_v = 0$, $deg_E(v) > 0$. Note that for no vertex, $x_v = 0$ since in this case all the neighbors if v have value 1

Let F be the set of edges whose rows are independent and $|F| = |V|$.

This gives a contradiction as any cycle implies that the rows of F are not independent. See Corollary 3 that implies that F is a forest, and so has at most $n - 1$ edges.

A polynomial algorithm for maximum matching in Bipartite graphs

The LP

$$
\begin{array}{rc}
\text{Maximize} & \sum_e y_e \cdot c_e \\
\text{Subject to:} & \\
 & \sum_{e \in E(v)} y_e \geq 0, \\
 & y_e \leq 1
\end{array}
$$

The algorithm we will show that works

The algorithm:

1. $F \leftarrow \emptyset$
2. **While** $E(G) \neq \emptyset$ **do:**
   (a) Compute a solution to the above LP.
   (b) Remove every edge e with $y_e = 0$.
   (c) If there is an edge $e = uv$ so that $y_e = 1$ then set $F \leftarrow F \cup F \cup \{e\}$ and $E \leftarrow E \setminus \{uv\}$
   (d) Remove all edges with at least one endpoint in e.
3. Return F

When is a graph a collection of cycles?

Claim 2 *If $deg(v) = 2$ for every v, then there is a collection of vertex disjoint cycles that contains all of V*

Proof. Consider an edge $e = uv$. Since both u, v have degree 1 now, but degree 2 in the graph, we can make the path longer by two, adding an edge to u and to v. In general we get a path whose first and last vertices have degree 1. Since the graph is finite, these two paths must meet getting a cycle. The same argument implies that G is a collection of cycles. □

Characterization of extreme points

Say we add $y_e \leq 1$.

There are more rows than columns so we need to choose a subset of the rows W so that $|W| = |E|$ because there are $|E|$ variables and so you need to choose $|E|$ rows. Note that we choose a set of $|E|$ **vertices**. WE should think of that as a set $W \subset V$.

Note that for the chosen rows the inequality also hold with **equality**. The vectors whose inequality is chosen to hold with equality Also, the vectors of the vertices must be **linearly independent**.

there are no cycles

Say that the graph restricted to the vertices in W with all the edges with both endpoints in W is a dependent set. If the cycle is $x_1 - -y_1 - -x_2 - -y_2 - -x_3 - -y_3$ for example, the vectors of x_1, x_2, x_3 have the same edges hence the same sum as the vectors of y_1, y_2, y_3.

This means that the graph $G(W)$ with W as vertices and edges with both endpoints in W is a **forest**.

Proof

Say that we have no edges that are 1 or 0. Namely for every edge $0 < y_e < 1$.

Let $deg_W(v)$ be the number of neighbors v has in W Its the degree of the vertex in $G(W)$ (we are assuming here that $v \in W$).

Claim 3 $deg_W(v) \geq 2$

Proof. We know by the characterization of an extreme point that that for every $v \in W$ $\sum_{uv,\ u \in W} y_u = 1$.

Since for $e = vu$, $y_{vu} = y_e < 1$ for every e, the degree $deg_W(v)$ of v in $G(W)$ is at least 2. □

Now we show its almost 2.

Claim 4 *Vertices in W have degree at most 2 in G. In particular vertices not in W have no edges to vertices in W.*

Proof. $2|W| = 2|E| = \sum_{v \in V} deg(v) \geq \sum_v deg_W(v) = 2|W|$

This means that all inequalities are equalities:

$2|W| = 2|E| = \sum_{v \in V} deg(v) = \sum_v deg_W(v) = 2|W|$

We know that $deg_W(v) \geq 2$. But if there exists a vertex that has a neighbor outside W then its degree is at least 3 as its degree inside W is 2. In this case we get contradiction if a degree 3 appears since we get $2|W| = 2|W| + 1$.

Thus vertices in $V - W$ have no edges to vertices in W. Also vertices have degree exactly 2 in $G(W)$. $\square$

Proof continued

By the claim above the graph $G(W)$ is a collection of cycles. Because the degrees are exactly 2 in $G(W)$. But we can not have cycles as it leads to linear dependance.

Thus there is an e so that $y_e = 0$ or $y_e = 1$.

This ends the proof.

One thing we proved was that Minimum Vertex cover is polynomial in bipartite graphs. Thus so is Maximum Independent Set. We shall later show that in bipartite graphs the max size matching are equal to the minimum size Vertex Cover. Thus can be proved by duality.