An Improved LP-based Approximation for Steiner Tree

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October 19, 2010

Outline

- Undirected Cut Relaxation
- Bidirected Cut Relaxation
- Directed Component Cut Relaxation
  - Bridge Lemma
  - Iterative Randomized Rounding

Steiner Tree Problem

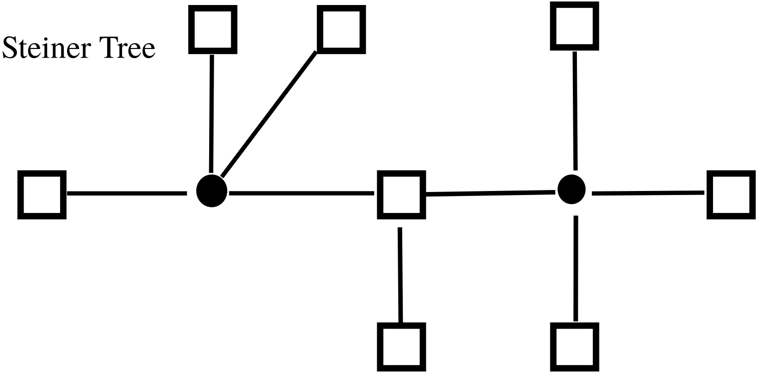
Input:

- Complete, Metric Graph $G = (V, E)$
- Weight of edges $w_e \qquad \forall e \in E$
- Set of terminals $R \subseteq V$

Output:

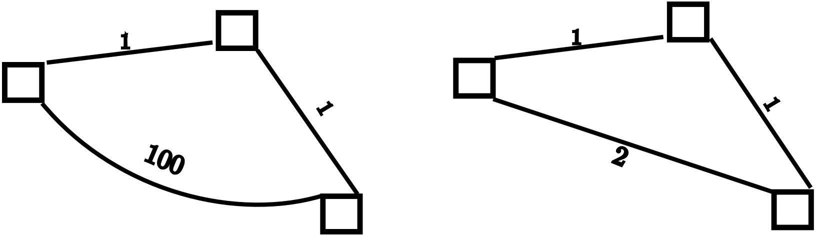
- Minimum cost tree which spans all the terminals in R

Steiner Tree Problem



Metric WLOG Image

Use shortest path metric



History

Hardness:

- NP hard even when edge costs$\in \{1,2\}$ [Bern & Plassmann' 89]
- APX Hard

Approximation:

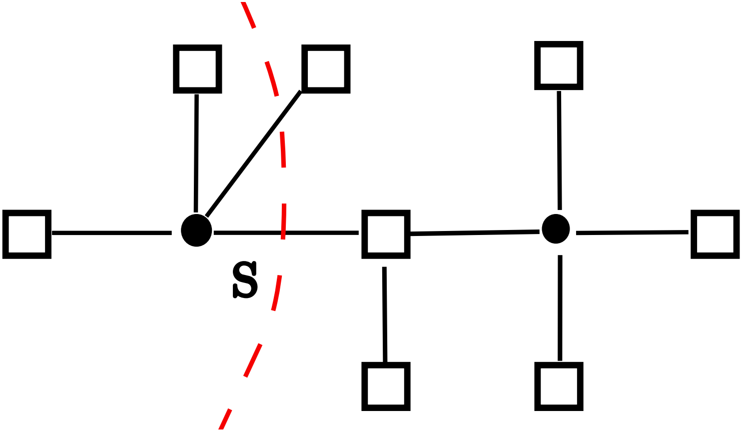
- 2-Apx (Minimum Spanning Tree)
- ...
- $1 + \frac{\ln 3}{2} + \varepsilon < 1.55$-Apx [Robins & Zelikovsky]

Linear Programs

Different Linear Programs:

- Undirected Cut Relaxation
- Bi-directed Cut Relaxation
- Directed Component Relaxation

Undirected Constraint



Undirected Cut Relaxation

Definition:

- $\delta(S)$:= edges crossing cut S

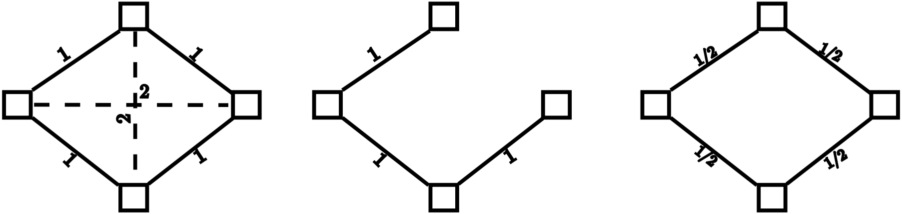
LP of UCR:

$$\min \sum_{e} w_e x_e$$

$$\sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall S = \{S \subset V \| u \in S, v \notin S\}$$

$$x_e \geq 0 \qquad \forall e \in E$$

Integrality Gap for UCR



Input $\quad$ $OPT_{IP} = n - 1 = 3$ $\quad$ $OPT_{LP} = \frac{n}{2} = 2$

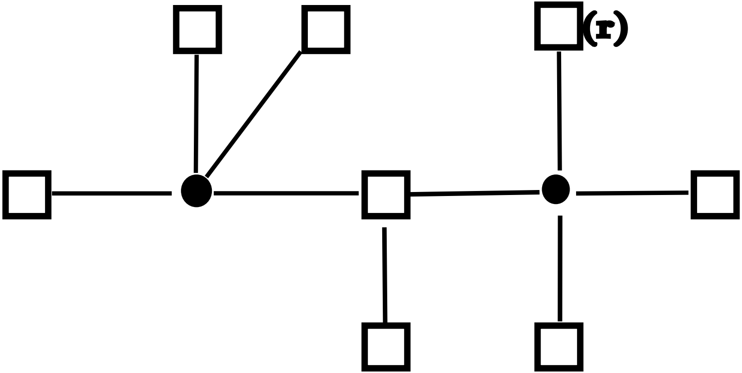
Integrality Gap = 2

Bi-directed Cut Relaxation

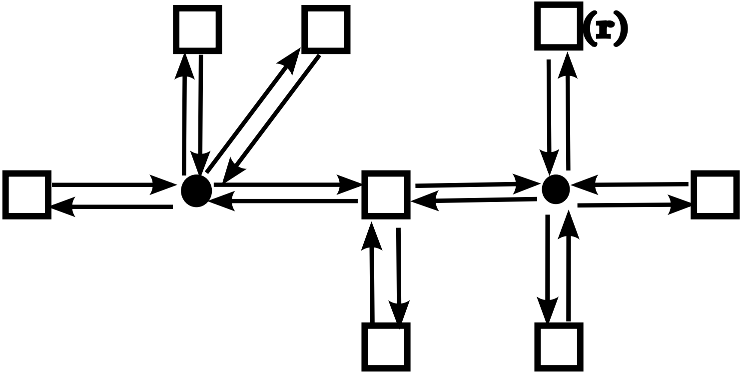
Definition:

- $\delta^+(S)$:= set of edges leaving S
- r := global root terminal

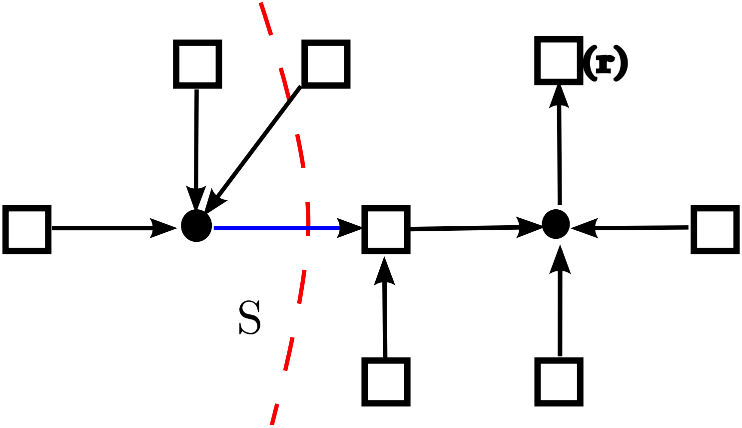
Bi-directed Cut Constraint



Bi-directed Cut Constraint



Bi-directed Cut Constraint



Bidirected Cut Relaxation

LP for BCR:

$$\min \sum_{e \in E} w_e x_e$$

$$\sum_{e \in \delta^+(S)} x_e \geq 1 \qquad \forall S \subseteq V \backslash \{r\},\ S \cap R \neq 0$$

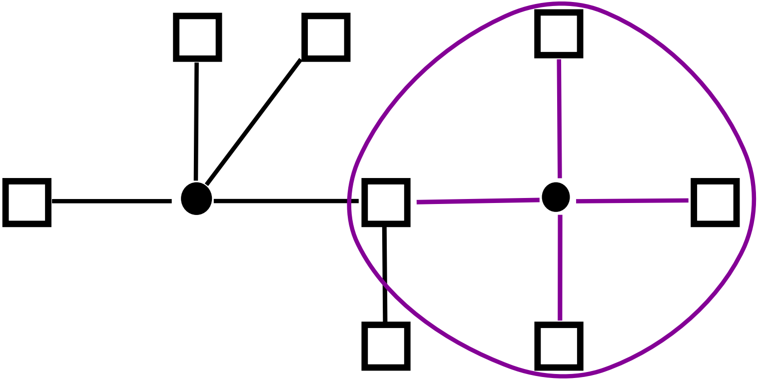
$$x_e \geq 0 \qquad \forall e \in E$$

theorem(Edmonds)

If $R = V \Rightarrow BCR\ is\ integral$

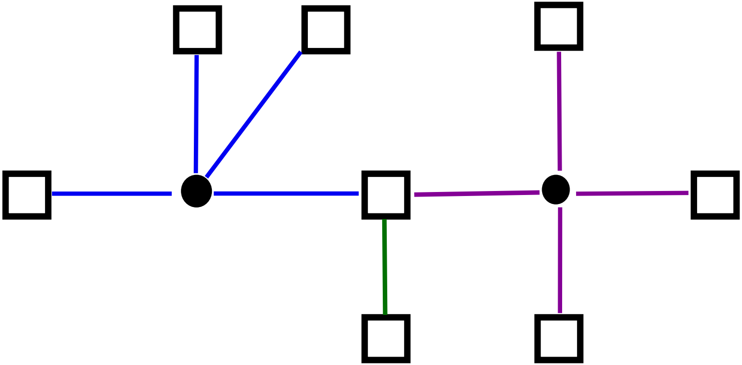
Component

A component is a maximal subtree whose terminals coincide with its leaves.



k-Restricted Steiner Tree

A k-restricted Steiner tree contains components with no more than k terminals



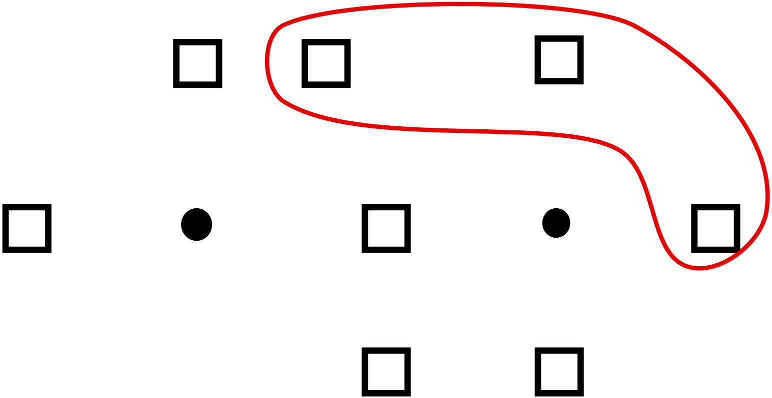
4-restricted Steiner Tree

Potential Components

We want to enumerate all possible components and determine their cost

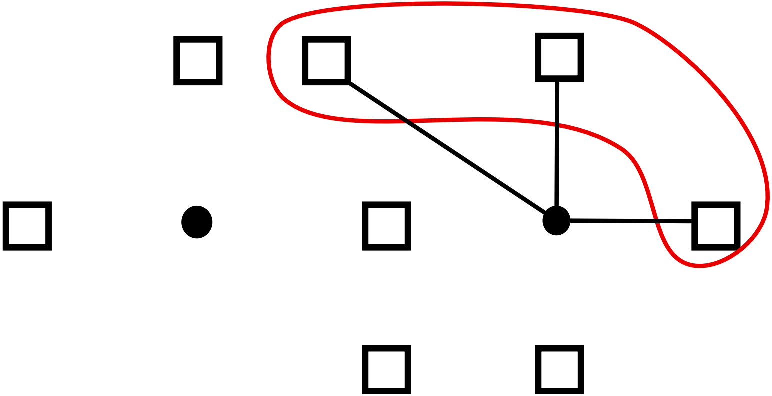
Potential Components

Choose arbitrary nodes $R(C_j)$ for component C_j



Potential Components

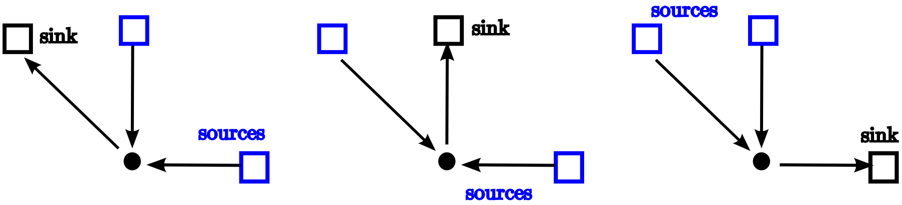
Cost of C_j = min Steiner tree on $R(C_j)$



Potential Directed Components

Definition :

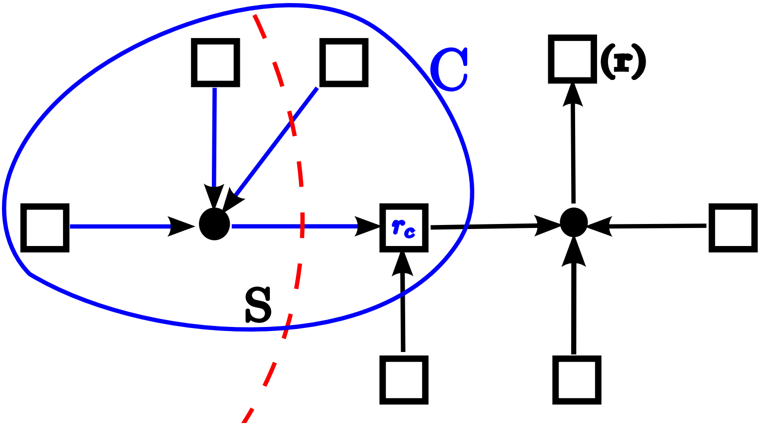
- $\text{sink}(C_j)$:= root terminal of C_j
- $\text{sources}(C_j) := R(C_j) \setminus \text{sink}(C_j)$



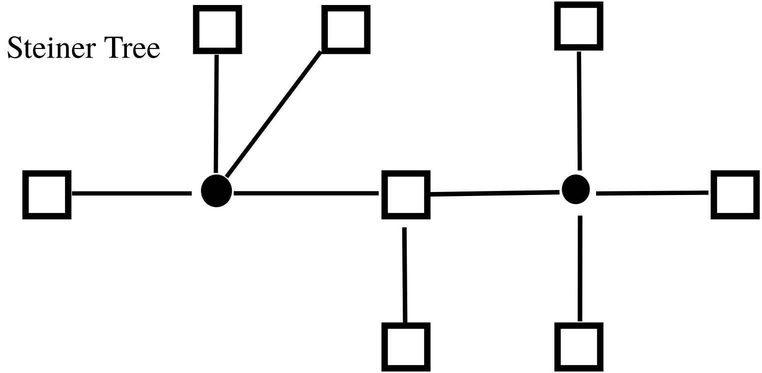
Direct all edges of the component to the sink node

Crossing Component

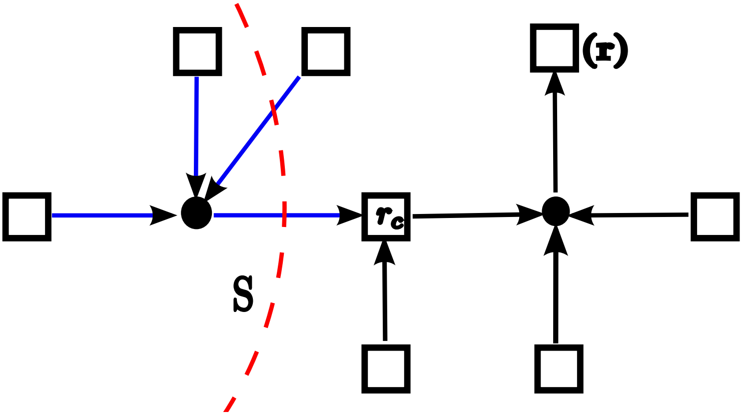
Component C crosses S when at least one source of $C \in S$ and $\text{sink}(C) \notin S$



LP for k-DCR



LP for k-DCR



LP for k-DCR

LP for k-DCR:

$$\min \sum_j c(C_j)x_j$$

$$\sum_{C_j \in \delta^+(S)} x_j \geq 1 \qquad \forall S \subseteq V \backslash \{r\},\ S \cap R \neq 0$$

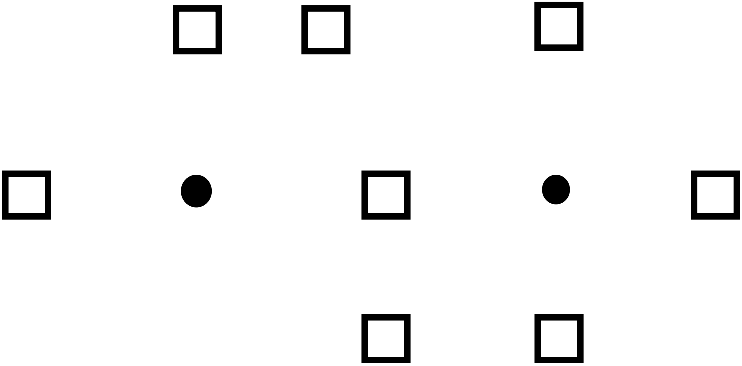
$$x_e \geq 0 \qquad \forall e \in E$$

Algorithm

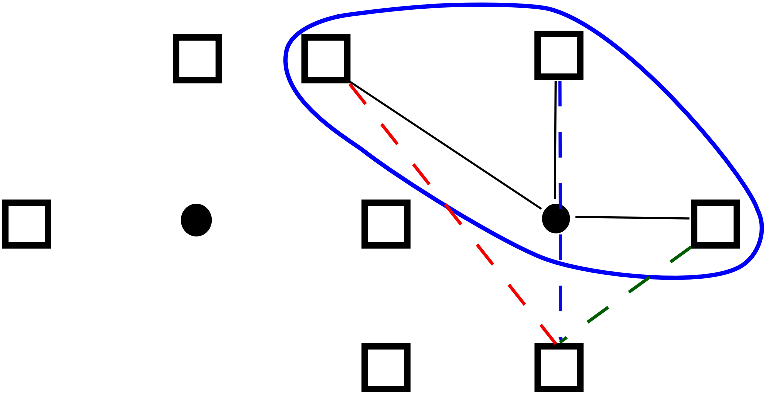
Iterative Randomized Rounding

- Solve LP-relaxation of the problem
- Select a component, C_j, with probability $\frac{x_j}{\sum x_j}$
- Shrink C_j to a single terminal
- Iterate the process on the residual problem

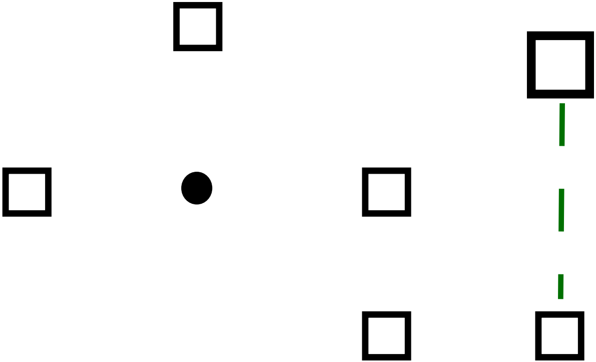
Algorithm



Algorithm



Algorithm



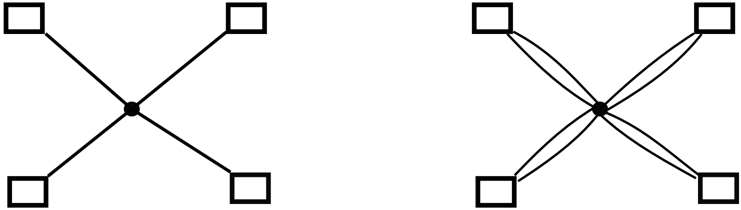
Algorithm

Algorithm

- For $t = 1, 2, 3 \ldots \mu$
  - Compute an optimal fractional solution x^t to k-DCR (w.r.t. the current instance)
  - Sample a component C^t, where $C^t = C_j$ with probability $\frac{{x_j}^t}{\sum {x_i}^t}$
  - Contract C^t
- Compute a terminal spanning tree T^μ on the remaining instance
- Output $T^\mu \cup \bigcup_{t=1}^{\mu} C^t$

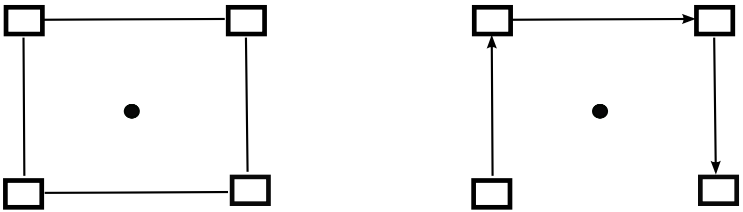
$C(T) \leq 2opt_k^f$

- Choose a component C_j with $x_j > 0$
- Double its edges to form a Eulerian tour
- Cost of Eulerian tour $\leq 2 \cdot c_j$



$C(T) \leq 2opt_k^f$

- Shortcut Eulerian tour to a TSP tour
- Delete one edge of TSP tour to make terminal spanning tree T_j
- Cost $T_j \leq 2 \cdot c_j$



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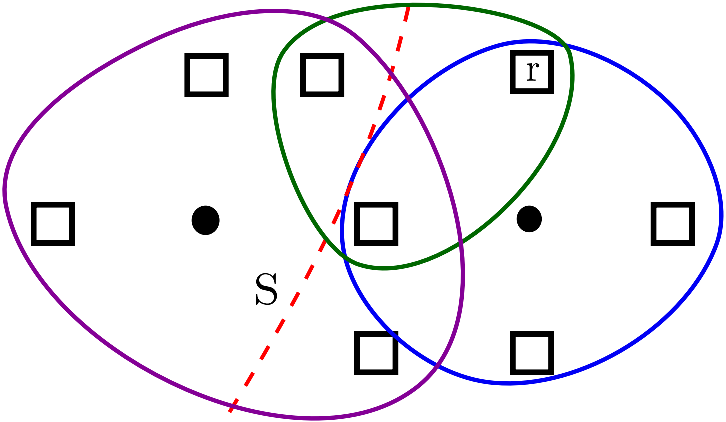

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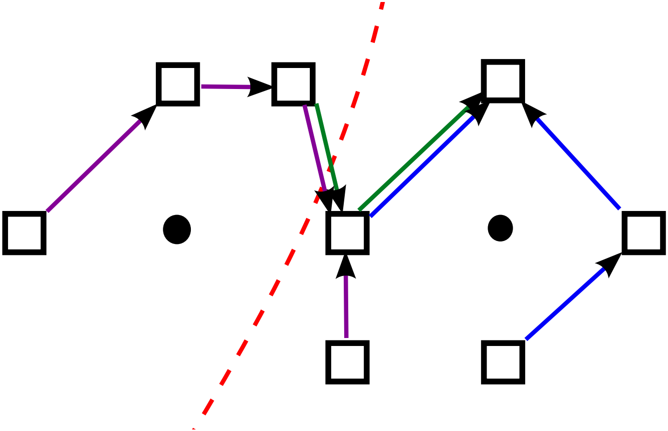
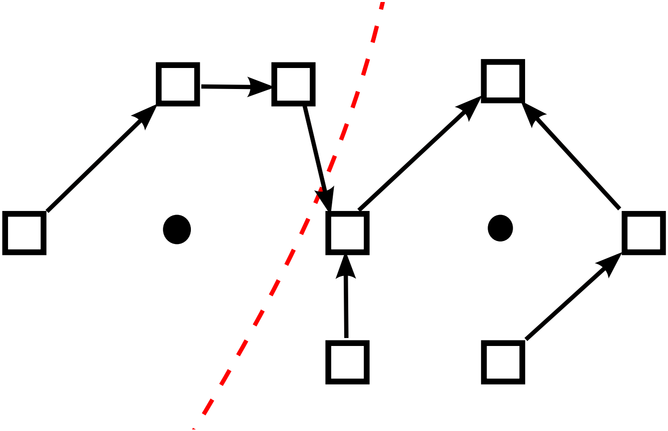


$C(T) \leq 2opt_k^f$



Bridges

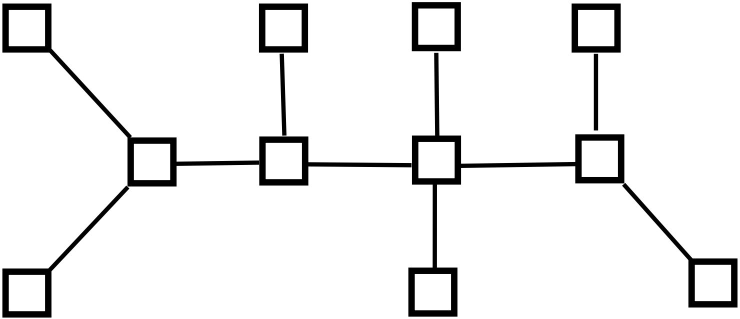
$Br_T(C_j)$:= bridges of tree T with respect to C_j
$br_T(C_j)$:= cost of edges in $Br_T(C_j)$

Finding $Br_T(C_j)$:

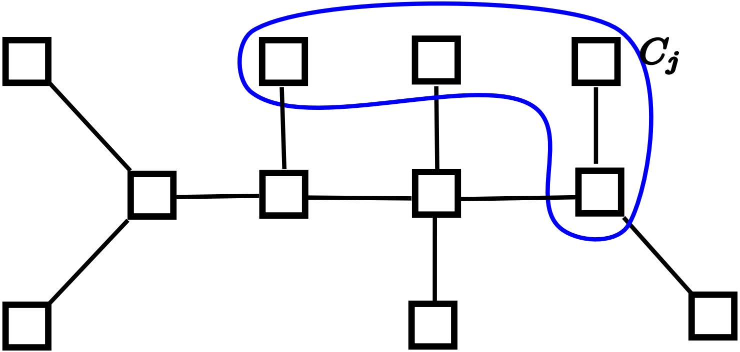
- contract C_j into a single vertex
- find minimum spanning subtree (of T) on new graph
- the bridges are all of the edges not in the subtree

Bridges

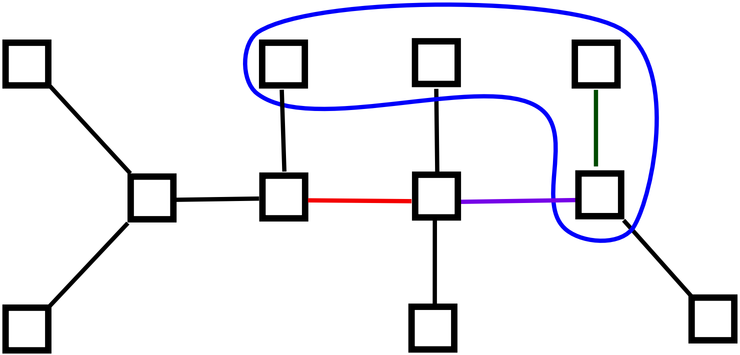
- Let T be a terminal spanning tree and C_j be one component



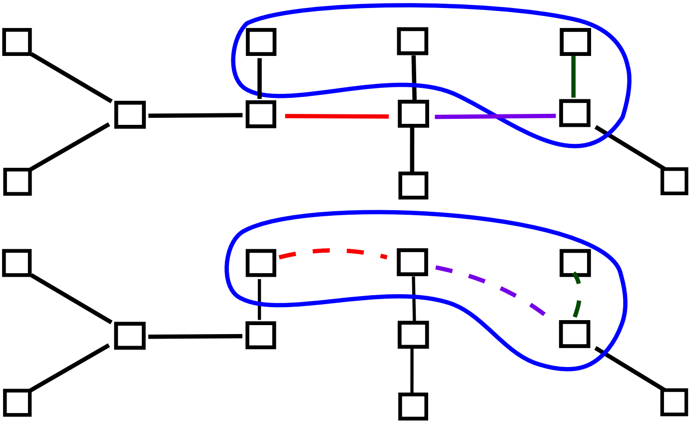
Bridges



Bridges



Bridges



- Reflecting the bridges

Statement of Bridge Lemma

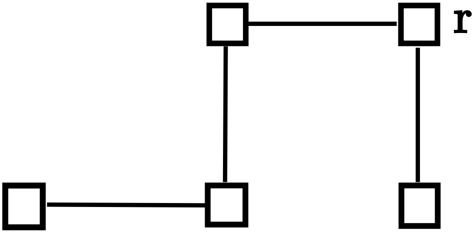
Theorem(Bridge Lemma)

Let T be a terminal spanning tree and x be a *k-DCR* solution. Then

$$c(T) \leq \sum_j x_j br_T(C_j)$$

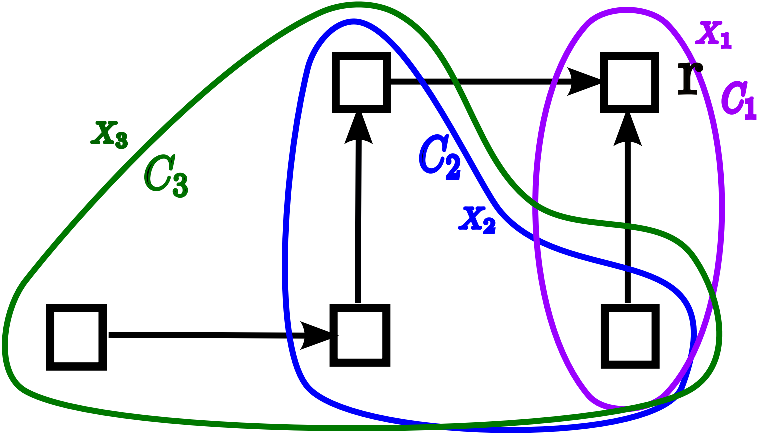
Statement of Bridge Lemma

Terminal spanning tree T



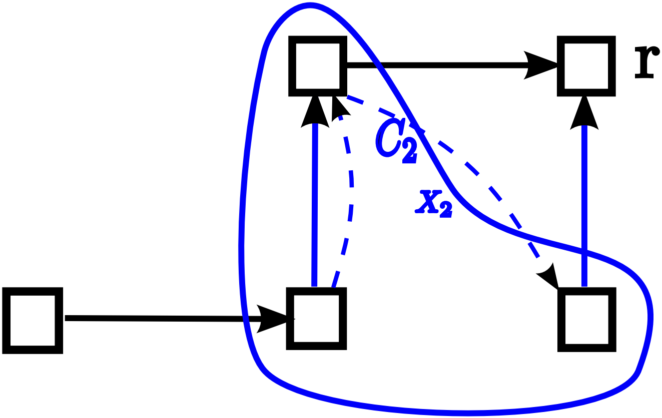
Statement of Bridge Lemma

LP solution and components C_j with $x_j > 0$



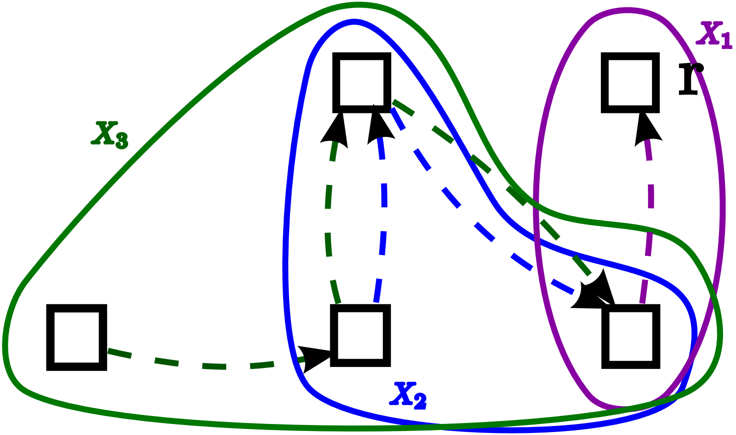
Statement of Bridge Lemma

Find $Br_T(C_j)$ and reflect them



Statement of Bridge Lemma

Find $Br_T(C_j)$ and reflect them

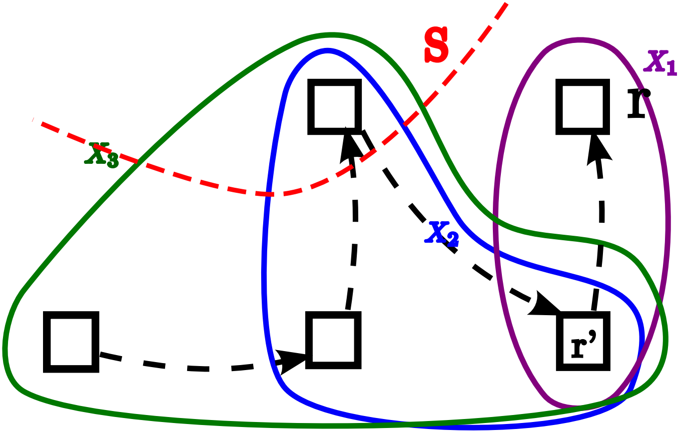


Statement of Bridge Lemma

- New terminal spanning tree Y
- $w(e)$ in Y is inherited from bridge weights
- Any spanning tree on Y costs more than T

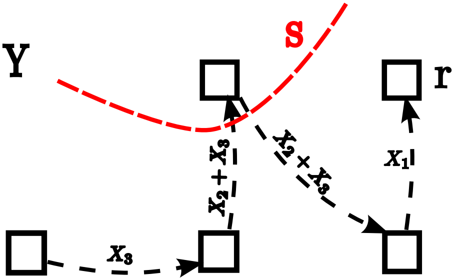
Statement of Bridge Lemma

- $x_2 + x_3 \geq 1$



Statement of Bridge Lemma

- $y(e) = \sum_{e \in C_j} x_j$
- $y(e)$'s are feasible solution for BCR



Bridge Lemma

theorem(Edmonds)

If $R = V \Rightarrow BCR$ *is integral*

There exist a integral terminal spanning tree F s.t.

$$w(F) \leq \sum_{e \in Y} w(e)y(e)$$

$$\sum_j x_j br_T(Cj) = \sum_j x_j w(Y_j) = \sum_{e \in Y} w(e)y(e) \geq w(F) \geq c(T)$$

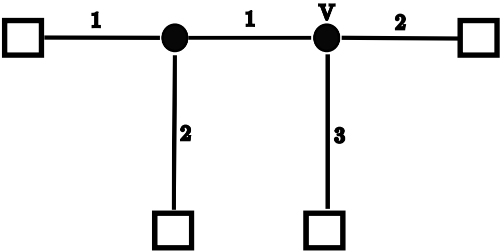
Bridge Lemma

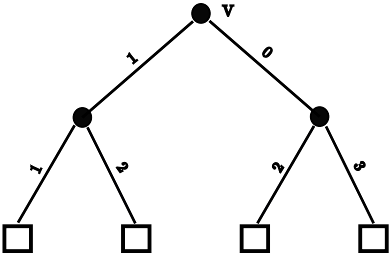
theorem(Bridge Lemma)

Let T be a terminal spanning tree and x be a *k-DCR* solution. Then

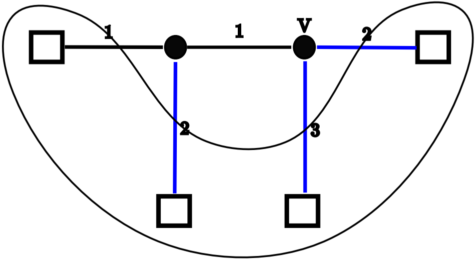
$$c(T) \leq \sum_j x_j br_T(C_j)$$

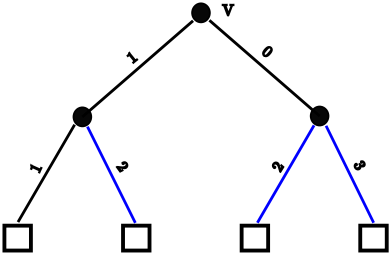
Bound on Bridges Weight



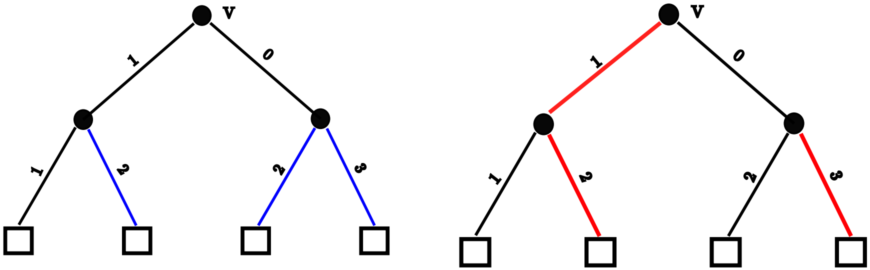


Bound on Bridges Weight





Bound on Bridges Weight



Algorithm

Algorithm

- For $t = 1, 2, 3...\mu$
  - Compute an optimal fractional solution x^t to k-DCR (w.r.t. the current instance)
  - Sample a component C^t, where $C^t = C_j$ with probability $\frac{x_j{}^t}{\sum x_i{}^t}$
  - Contract C^t
- Compute a terminal spanning tree T^μ on the remaining instance
- Output $T^\mu \cup \bigcup_{t=1}^{\mu} C^t$

Improved Approximation for Steiner Tree

Slightly different LP

- $\sum x_j$ might vary through iterations.
- Add a dummy component C containing only r
- $c(C) = 0$
- Add the constraint $\sum x_j = \Sigma$
- Σ is an upper bound on $\sum x_j$

Improved Approximation for Steiner Tree

New LP for k-DCR:

$$\min \sum_j c(C_j)x_j$$

$$\sum_{C_j \in \delta^+(S)} x_j \geq 1 \qquad \forall S \subseteq V - r$$

$$x_e \geq 0 \qquad \forall e \in E$$

$$\sum_j x_j = \Sigma$$

Bound T^{μ}

Lemma

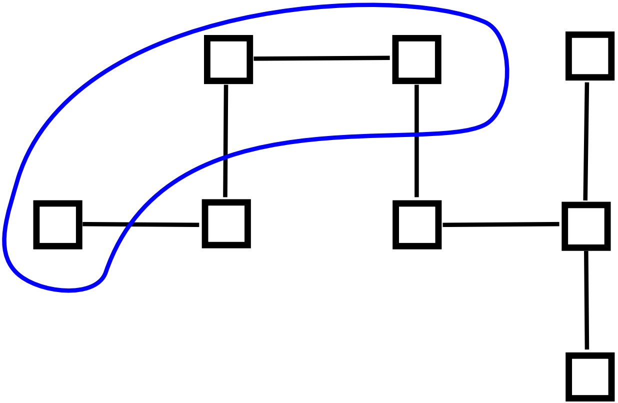
$E[c(T^{\mu})] \leq 2(1 - \frac{1}{\Sigma})^{\mu} opt_k^f$

Bound T^μ

$$E[c(T^t)] \leq c(T^{t-1}) - E[br_T^{t-1}(C^t)]$$

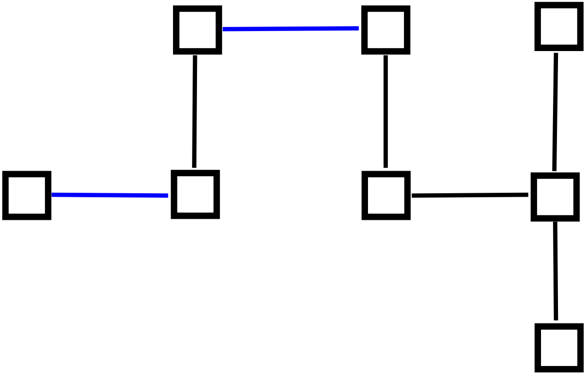
Bound T^μ

MST on terminals and potential component



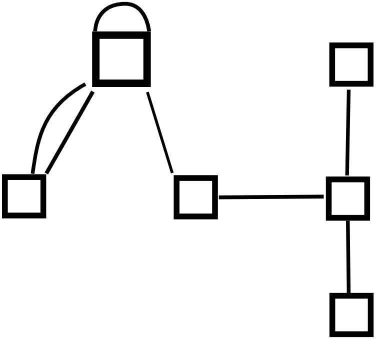
Bound T^μ

Bridges for this component on tree T

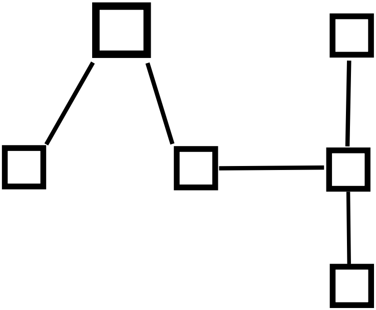


Bound T^{μ}

- Bridges form cycles in new MST



- Delete bridges we have new MST



Bound T^{μ}

Lemma

$E[c(T^{\mu})] \leq 2(1 - \frac{1}{\Sigma})^{\mu} opt_k{}^f$

$$
\begin{aligned}
E[c(T^t)] &\leq c(T^{t-1}) - E[br_T^{t-1}(C^t)] \\
&= c(T^{t-1}) - \frac{1}{\Sigma} \sum_j x_j{}^t br_T^{t-1}(C_j) \\
&\leq (1 - \frac{1}{\Sigma}) c(T^{t-1}) && \text{(Bridge Lemma)}
\end{aligned}
$$

$$E[c(T^{\mu})] \leq (1 - \tfrac{1}{\Sigma})^{\mu} c(T^0) \leq 2(1 - \tfrac{1}{\Sigma})^{\mu} opt_k{}^f$$

Bound Component

Lemma

Let S^t be the optimal steiner tree after t iterations ,then we have

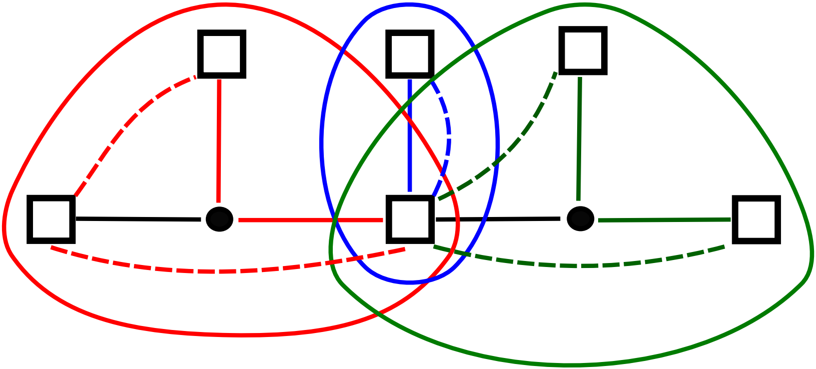
$$E[c(S^t)] \leq (1 - \frac{1}{2\Sigma})^t c(S)$$

Bound Component

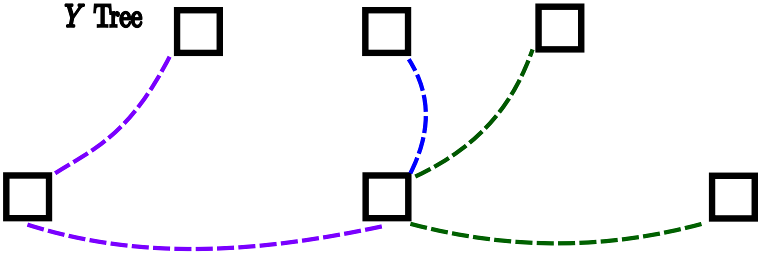
$$
\begin{aligned}
E[c(S^t)] &= c(S^{t-1}) - E[c(C_j)] \\
&\leq c(S^{t-1}) - E[br_{S^{t-1}}(C_j)]
\end{aligned}
$$

Bound Component

Reflect Y tree

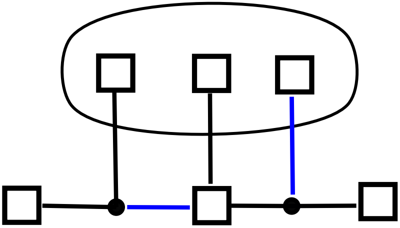


Bound Component

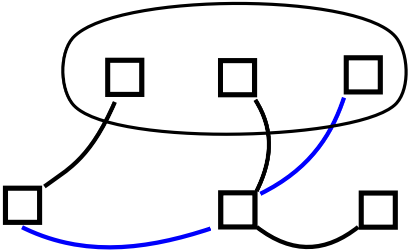


Bound Component

- Bridges based on Steiner tree



- Bridges based on Y tree



Bound Component

Lemma

Let S^t be the optimal steiner tree after t iterations ,then we have

$$E[c(S^t)] \leq (1 - \frac{1}{2\Sigma})^t c(S)$$

Bound Component

$$
\begin{aligned}
E[c(S^t)] &= c(S^{t-1}) - E[c(C_j)] \\
&\leq c(S^{t-1}) - E[br_{S^{t-1}}(C_j)] \\
&\leq c(S^{t-1}) - E[br_Y(C_j)] \\
&= c(S^{t-1}) - \frac{1}{\Sigma} \sum_j x_j br_Y(C_j) \\
&\leq c(S^{t-1}) - \frac{1}{\Sigma} w(Y) \\
&\leq c(S^{t-1}) - \frac{1}{2\Sigma} c(S^{t-1}) \\
&= (1 - \frac{1}{\Sigma}) c(S^{t-1})
\end{aligned}
$$

Bound Component

Lemma

Final $c(S^t)$ relates to steiner tree before first iteration

$$E[c(S^t)] \leq (1 - \frac{1}{2\Sigma})^{t-1} c(S)$$

Bound Component

Corollary

For every $t = 1, ..., \mu, E[c(C^t)] \leq \frac{1}{\Sigma}(1 - \frac{1}{2\Sigma})^{t-1} opt_k$

$$
\begin{aligned}
E[c(C^t)] &\leq \frac{1}{\Sigma} E[\sum_j (x_j)^t c(C_j)] \\
&= \frac{1}{\Sigma} E[(opt_k)^{f,t}] \\
&= \frac{1}{\Sigma} E[(opt_k)^t] \\
&\leq (1 - \frac{1}{2\Sigma})^{t-1} opt_k
\end{aligned}
$$

Final Result

Lemma

For any $k = O(1)$, there is a polynomial-time expected $\frac{1}{2}$-approximation algorithm for k-restricted Steiner tree

$$
\begin{aligned}
E\left[\frac{c(T^{\mu}) + \sum_{t=1}^{\mu} c(C^t)}{c(S)}\right] &\leq 2(1-\frac{1}{\Sigma})^{\mu} + \frac{1}{\Sigma}\sum(1-\frac{1}{2\Sigma})^{t-1} \\
&= 2(1-\frac{1}{\Sigma})^{\delta\Sigma} + 2 - 2(1-\frac{1}{2\Sigma})^{\delta\Sigma} \\
&\leq 2e^{-\delta} + 2 - 2e^{\frac{-\delta}{2}} \\
&= \frac{3}{2}
\end{aligned}
$$