Partial Vertex Cover

Joshua Wetzel

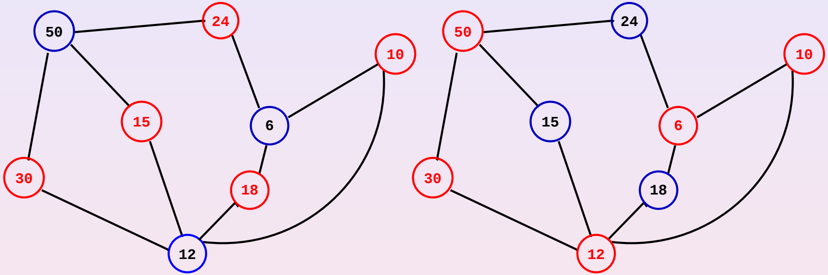
Department of Computer Science
Rutgers University–Camden
wetzeljo@camden.rutgers.edu

April 2, 2009

Vertex Cover

- **Input:**
  - Given $G = (V, E)$
  - Non-negative weights on vertices
- **Objective:**
  - Find a least-weight collection of vertices such that each edge in G in incident on at least one vertex in the collection

Vertex Cover: Example



$COST = 97$ $COST = 108$

Vertex Cover: IP Formulation

$x_v \leftarrow 1$ if v is in our cover, 0 otherwise

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v x_v \\ \text{s.t.} \quad & \\ & x_a + x_b \geq 1, \;\; \forall \;\; e = (a, b) \\ & x_v \in \{0, 1\}, \;\; \forall \;\; v \in V \end{aligned}$$

LP Relaxation

- Integer programs have been shown to be NP-hard
- Relax the integrality constraints $x_v \in \{0,1\}, \ \forall \ v \in V$

$$
\begin{aligned}
\min \quad & \sum_{v \in V} w_v x_v \\
\text{s.t.} \quad & \\
& x_a + x_b \geq 1, \ \forall \ e = (a,b) \\
& x_v \geq 0, \ \forall \ v \in V
\end{aligned}
$$

Constructing the Dual LP

Primal LP:

$$\begin{aligned} \min \quad & \sum_{v \in V} w_v x_v \\ \text{s.t.} \quad & \\ & x_a + x_b \geq 1, \ \forall \ e = (a, b) \qquad (y_e) \\ & x_v \geq 0, \ \forall \ v \in V \end{aligned}$$

Primal LP and Dual LP

Primal LP:

$$\min \sum_{v \in V} w_v x_v$$

s.t.

$$x_a + x_b \geq 1, \ \forall \ e = (a, b)$$
$$x_v \geq 0, \ \forall \ v \in V$$

Dual LP:

$$\max \sum_{e \in E} y_e$$

s.t.

$$\sum_{e \,:\, e \text{ hits } v} y_e \leq w_v, \ \forall \ v \in V$$
$$y_e \geq 0, \ \forall \ e \in E$$

Primal-Dual Method

$$\boxed{Dual_{Feasible} \leq Dual_{OPT} = Primal_{OPT}}$$

$$\boxed{Primal_{OPT} \leq OPT_{IP}}$$

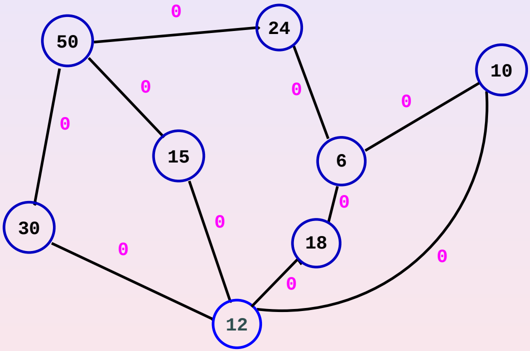
- Construct the dual LP
- Construct an algorithm that manually tightens dual constraints to obtain a 'maximal' dual solution

Clarkson's Algorithm

- Inititally all edges are uncovered.
- While $\exists$ an uncovered edge in G:
  - raise y_e for all uncovered edges simultaneouslty until a vertex, v, becomes full (i.e $\sum_{e:e \text{ hits } v} y_e = w_v$)
  - $C \leftarrow C \cup \{v\}$
  - any e that touches v is covered.
- Return C as our vertex cover

Clarkson's Algorithm: Example

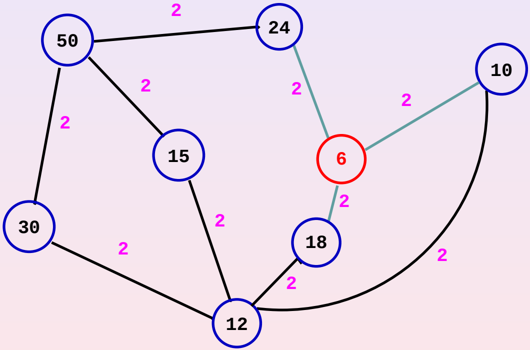
$$\sum_{e:e \text{ hits } v} y_e \leq w_v$$



- Raise each y_e uniformly until a vertex is full.

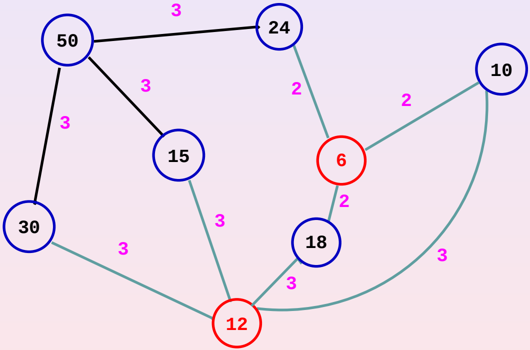
Clarkson's Algorithm: Example

$$\sum_{e:e \text{ hits } v} y_e \leq w_v$$



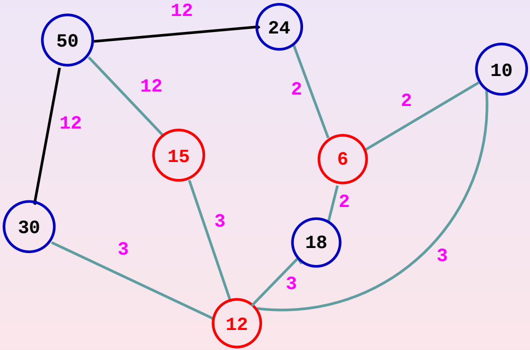
Clarkson's Algorithm: Example

$$\sum_{e:e \text{ hits } v} y_e \leq w_v$$



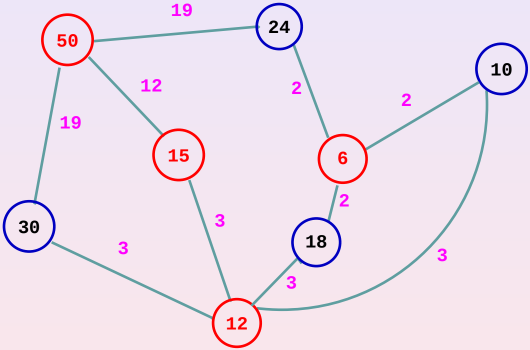
Clarkson's Algorithm: Example

$$\sum_{e:e \text{ hits } v} y_e \leq w_v$$

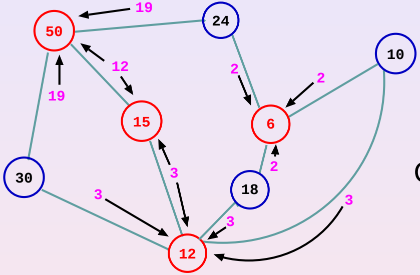


Clarkson's Algorithm: Example

$$\sum_{e:e \text{ hits } v} y_e \leq w_v$$



COST = 83



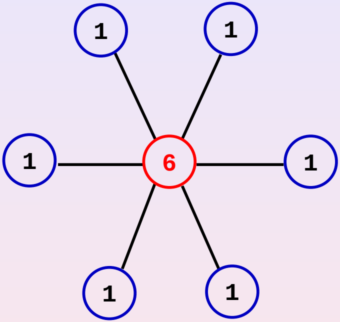
$$\sum_{e:e \text{ hits } v} y_e \leq w_v$$

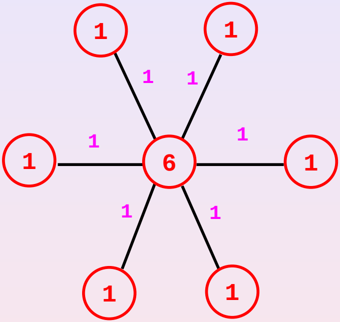
Dual Obj. Fn:

$$\max \sum_{e} y_e$$

$$\begin{aligned}
\text{Our Cost} &= \text{wt(red vertices)} \\
&\leq 2 \sum_{e \text{ hits red}} y_e \\
&\leq 2 \sum_{e} y_e \\
&= 2DFS \\
&\leq 2OPT
\end{aligned}$$

Clarkson's Algorithm: Tight Example





$COST_{OPT} = 6$

$COST_{Clarkson} = 12$

Partial Vertex Cover

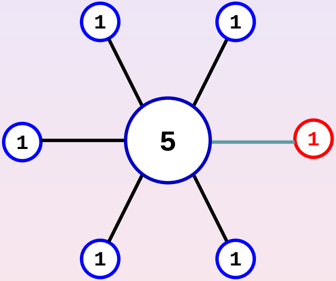
- **Input:**
  - Graph, $G = (V, E)$
  - Non-negative integer weights for vertices
  - Integer, k
- **Objective:**
  - Find the least cost set of vertices in G that will cover at least k edges.

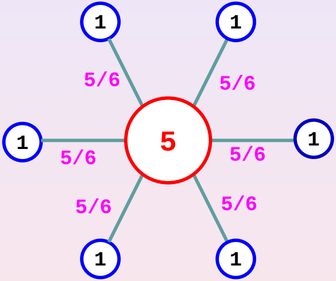
Key Issue

- In full vertex cover OPT covers all edges and hence we know which edges to cover.
- In partial vertex cover, we do not know which k edges OPT covers.
- When k is part of the input, the techniques for full-coverage do not directly apply.

Clarkson's Algorithm Fails

Input: $k = 1$





$COST_{OPT} = 1$

$COST_{Clarkson} = 5$

Related Work

- **Bshouty & Burroughs 1998:** solve the LP, modify it and then round the modified solution. 2-approximation for partial vertex cover.
- **Hochbaum 1998:** 2-approximation for partial vertex cover.
- **Bar-Yehuda 1999:** “local ratio” method, 2-approximation for partial vertex cover.
- **Mestre 2005:** 2-approximation primal-dual algorithm for partial vertex cover with improved running time

Vertex Cover: IP Formulation

$x_v \leftarrow 1$ if v is in our cover, 0 otherwise

$$
\begin{aligned}
\min \quad & \sum_{v \in V} w_v x_v \\
\text{s.t.} \quad & \\
& x_a + x_b \geq 1, \ \forall \ e = (a, b) \\
& x_v \in \{0, 1\}, \ \forall \ v \in V
\end{aligned}
$$

Partial Vertex Cover: IP formulation

$x_v \leftarrow 1$ if vertex v is chosen in the cover, 0 otherwise.
$y_e \leftarrow 1$ if edge e is *not* covered, 0 otherwise.

$$
\begin{aligned}
\min \quad & \sum_{v \in V} w_v x_v \\
\text{s.t.} \quad & \\
& y_e + x_a + x_b \geq 1, \forall e = (a, b) \\
& \sum_{e \in E} y_e \leq m - k \\
& x_v \in \{0, 1\}, \quad \forall v \in V \\
& y_e \in \{0, 1\}, \quad \forall e \in E
\end{aligned}
$$

Patial Vertex Cover: LP formulation

- Relax the integrality constraints.

$$x_v \in \{0,1\} \rightarrow x_v \geq 0, \ \ \forall \ \ v \in V$$
$$y_e \in \{0,1\} \rightarrow y_e \geq 0, \ \ \forall \ \ e \in E$$

Constructing the Dual LP

$$
\begin{aligned}
\min \quad & \sum_{v \in V} w_v x_v \\
\text{s.t.} \quad & \\
& y_e + x_a + x_b \geq 1, \;\; \forall \;\; e = (a, b) \qquad\qquad (u_e) \\
& \sum_{e \in E} y_e \leq (m - k) \qquad\qquad\qquad\qquad\qquad (z) \\
& x_v \geq 0, \;\; \forall \;\; v \in V \\
& y_e \geq 0, \;\; \forall \;\; e \in E
\end{aligned}
$$

Primal LP and Dual LP

Primal LP:

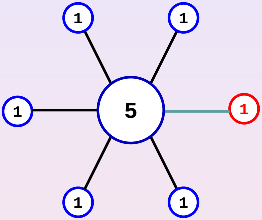
$$\begin{aligned}
\min \quad & \sum_{v \in V} w_v x_v \\
\text{s.t.} \quad & \\
& y_e + x_a + x_b \geq 1, \quad \forall e \\
& \sum_{e \in E} y_e \leq m - k \\
& x_v \geq 0, \quad \forall \; v \in V \\
& y_e \geq 0, \quad \forall \; e \in E
\end{aligned}$$

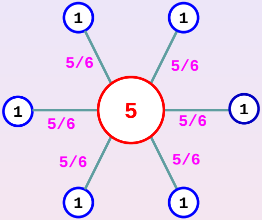
Dual LP:

$$\begin{aligned}
\max \quad & \sum_{e \in E} u_e - z(m-k) \\
\text{s.t.} \quad & \\
& \sum_{e \,:\, e \text{ hits } v} u_e \leq w_v, \forall v \in V \\
& u_e \leq z, \quad \forall \; e \in E \\
& u_e \geq 0, \quad \forall \; e \in E \\
& z \geq 0
\end{aligned}$$

Intuition for Primal-Dual

Input: $k = 1$





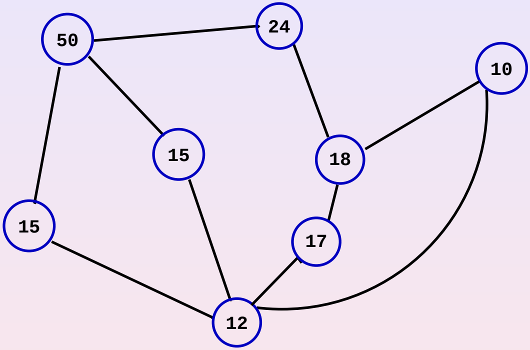
$COST_{OPT} = 1$ $COST_{Clarkson} = 5$

- In last iteration, Clarkson's Alg. may choose more edges than we need at a very high cost
- We must somehow bound the cost of last vertex chosen

Primal-Dual Algorithm

- For each vertex, v, in G
  - Guess v to be the heaviest vertex in OPT, called v_h
  - $C_v \leftarrow \{v_h\}$
  - Raise weight of all heavier vertices in G to ∞
  - Remove all edges touching v_h from G
  - $k' \leftarrow k - deg(v_h)$
  - Run Clarkson on this instance until k' edges are covered.
  - $C_{choices} \leftarrow C_{choices} \cup C_V$
- Return the lowest cost cover, C

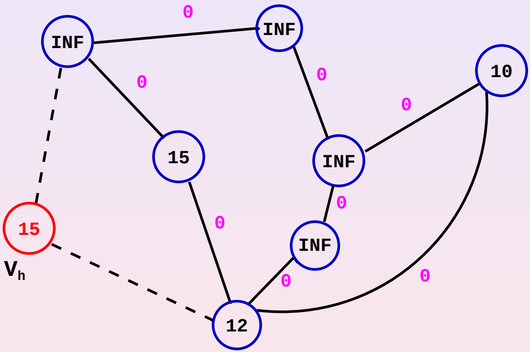
Primal-Dual Algorithm: Example



Input: $G = (V, E)$, $\quad k = 8$

Primal-Dual Algorithm: Example

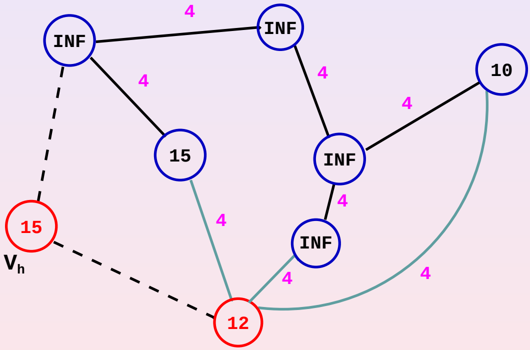
$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



$k' = 6$

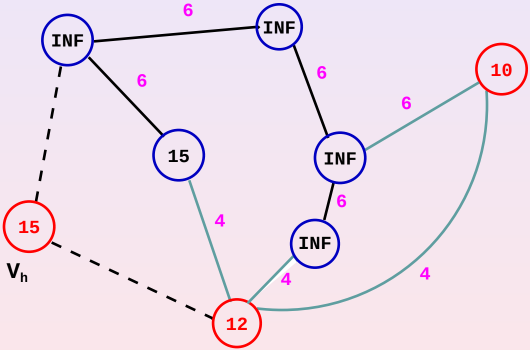
Primal-Dual Algorithm: Example

$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



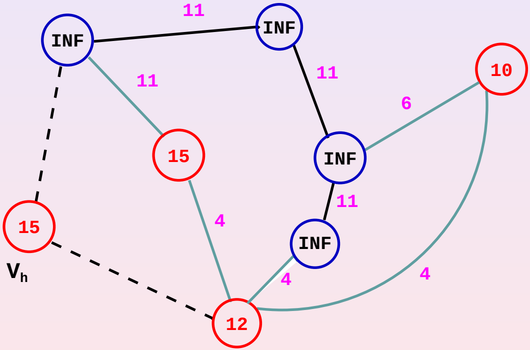
Primal-Dual Algorithm: Example

$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



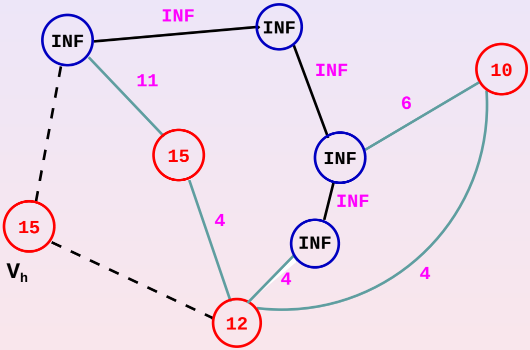
Primal-Dual Algorithm: Example

$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



Primal-Dual Algorithm: Example

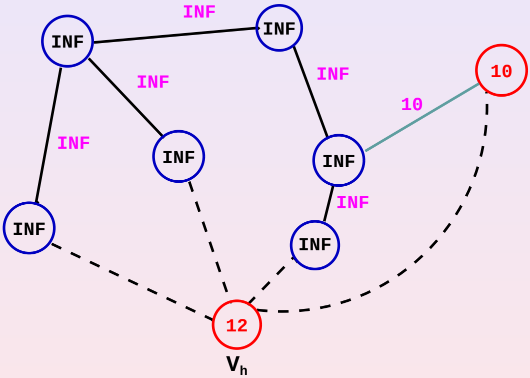
$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



$$COST = \infty$$

Primal-Dual Algorithm: Example

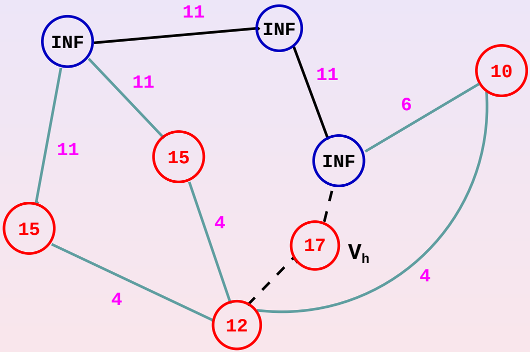
$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



$$COST = \infty$$

Primal-Dual Algorithm: Example

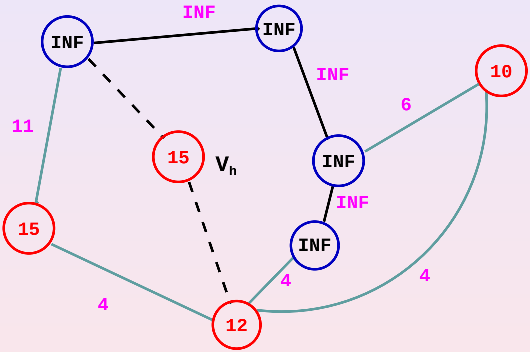
$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



$COST = 69$

Primal-Dual Algorithm: Example

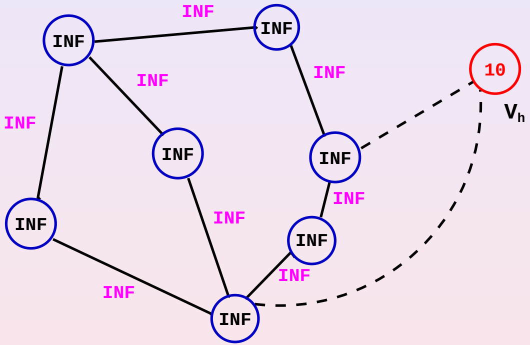
$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



$COST = \infty$

Primal-Dual Algorithm: Example

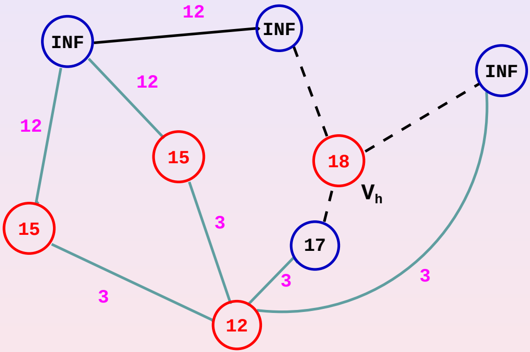
$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



$COST = \infty$

Primal-Dual Algorithm: Example

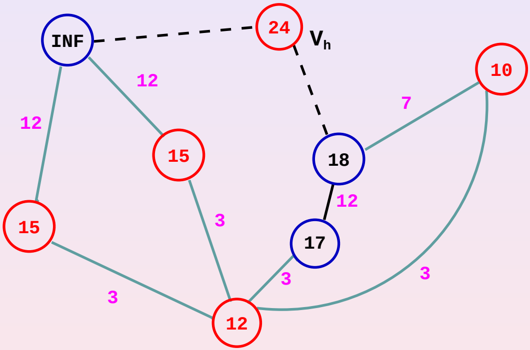
$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



$COST = 60$

Primal-Dual Algorithm: Example

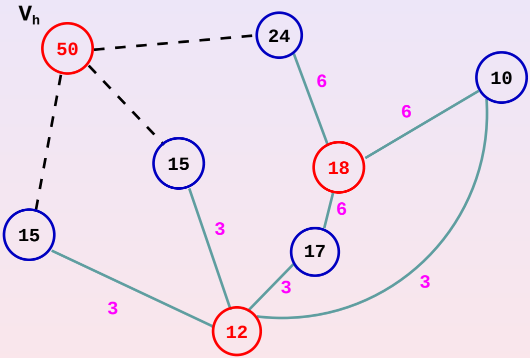
$$\sum_{e:e \text{ hits } v} u_e \leq w_v$$



$COST = 76$

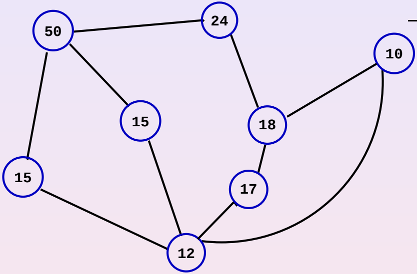
Primal-Dual Algorithm: Example

$$\sum_{e: e \text{ hits } v} u_e \leq w_v$$



$COST = 80$

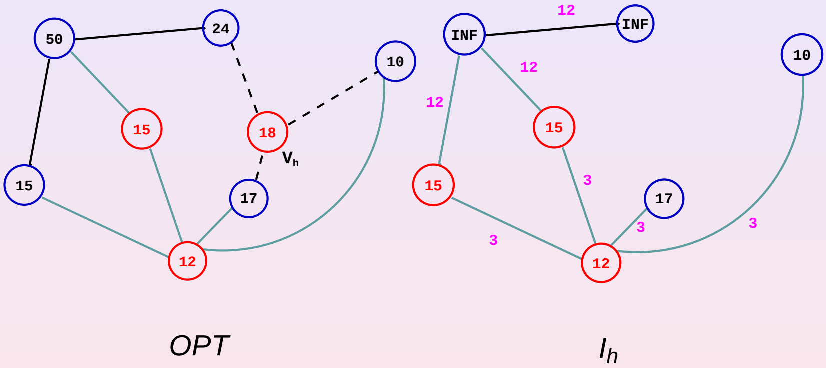
Primal-Dual Algorithm: Example



Heaviest Vertex	Cost
15	∞
12	∞
17	69
15	∞
10	∞
18	60
24	76
50	80

- Return cover with cost of 60

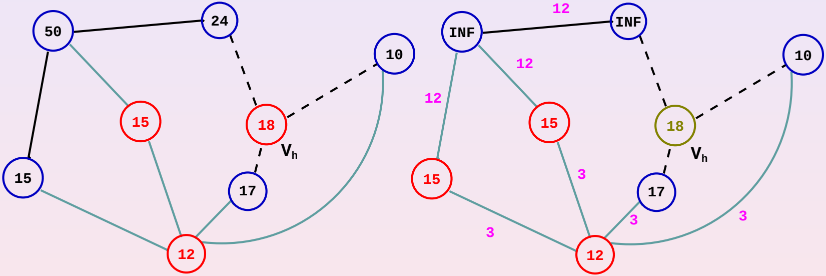
- I_h: inst. in which we correctly guess heaviest vertex in OPT



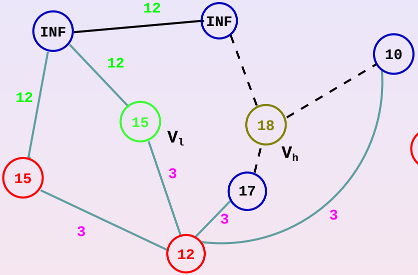
$$\boxed{OPT = OPT(I_h) + w(v_h)}$$

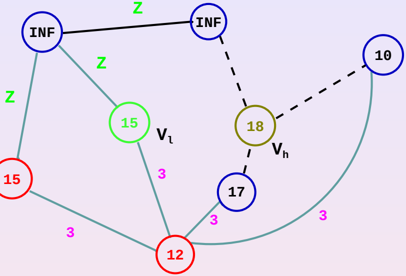
Analysis: Our Cost

$$\boxed{OPT = OPT(I_h) + w(v_h)}$$



$OPT(I_h) + w(v_h) = 45$ $\quad$ $COST(I_h) + w(v_h) = 60$





I_h

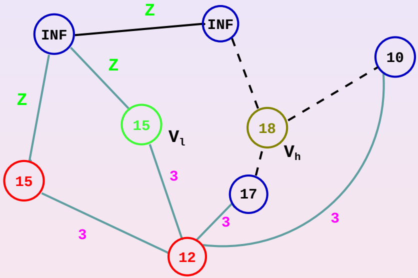
$$u_e \leq z, \;\; \forall \;\; e$$

$$z = 12$$

- Num of edges for which $u_e = z$ is at least $m' - k'$

Analysis: Our Cost

$$\boxed{OPT = OPT(I_h) + w(v_h)}$$



I_h

$$
\begin{aligned}
& \text{Our Cost} \leq Cost(I_h) + w(v_h) \\
= \; & w(RedVert) + w(v_l) + w(v_h) \\
\leq \; & w(RedVert) + 2w(v_h) \\
\leq \; & 2 \sum_{e \text{ hits red}} u_e + 2w(v_h) \\
\leq \; & 2[\sum_e u_e - z(m' - k')] + 2w(v_h) \\
\leq \; & 2DFS(I_h) + 2w(v_h) \\
\leq \; & 2OPT(I_h) + 2w(v_h) \\
\leq \; & 2OPT
\end{aligned}
$$

Primal LP and Dual LP

Primal LP:

$$\min \sum_{v \in V} w_v x_v$$

s.t.

$$y_e + x_a + x_b \geq 1, \quad \forall e$$

$$\sum_{e \in E} y_e \leq m - k$$

$$x_v \geq 0, \quad \forall \ v \in V$$

$$y_e \geq 0, \quad \forall \ e \in E$$

Dual LP:

$$\max \sum_{e \in E} u_e - z(m-k)$$

s.t.

$$\sum_{e \,:\, e \text{ hits } v} u_e \leq w_v, \forall v \in V$$

$$u_e \leq z, \quad \forall \ e \in E$$

$$u_e \geq 0, \quad \forall \ e \in E$$

$$z \geq 0$$

Reference

- **K. L. Clarkson.** A modification of the greedy algorithm for the vertex cover. *Information Processing Letters* 16:23-25, 1983.
- **R. Gandhi, S. Khuller, and A. Srinivasan.** Approximation Algorithms for Partial Covering Problems. Journal of Algorithms, 53(1):55-84, October 2004.

Thank You.