Multiway Cut

Sean Lowen

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Problem

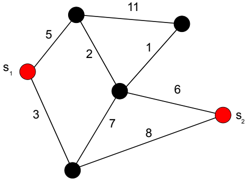
Input

An undirected graph $G = (V, E)$ with

- Non-negative weights w_e, $\forall e \in E$
- Set of terminals $S = \{s_1, s_2, \cdots, s_k\} \subseteq V$

Output

Subset of edges at min cost whose removal disconnects all terminals from each other



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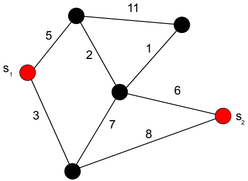

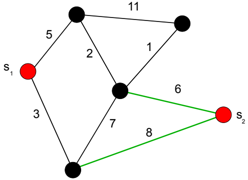

Cost $= 14$

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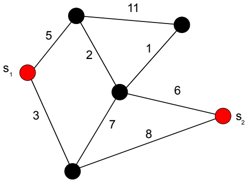
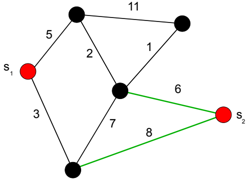
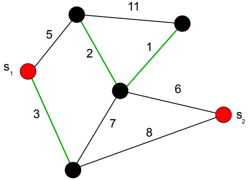




Cost = 14



Cost = 6

Outline

Algorithm

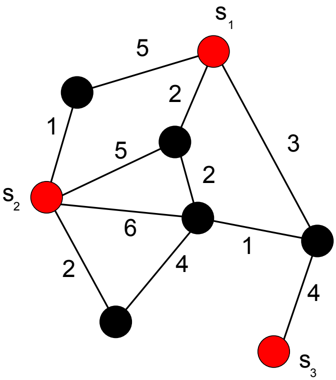
Multiway Cut Algorithm

1. $S = \{s_1, \cdots, s_k\}$
2. **for** $i \leftarrow 1$ to k **do**:

   $t_i \leftarrow$ contraction of $S \setminus \{s_i\}$

   $Z_i \leftarrow$ min s_i-t_i cut
3. let $w(Z_1) \leq w(Z_2) \leq \cdots \leq w(Z_k)$
4. **return** $\bigcup_{i=1}^{k-1} Z_i$

2-2/k Approx Example



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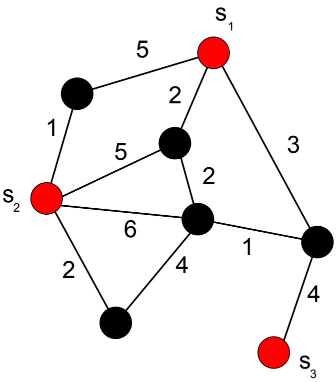

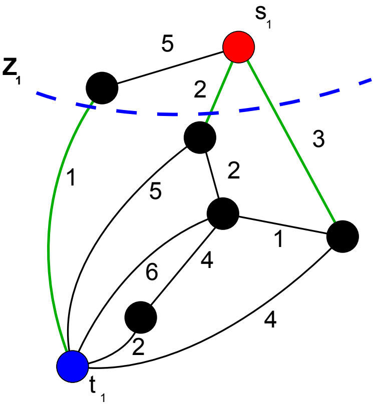

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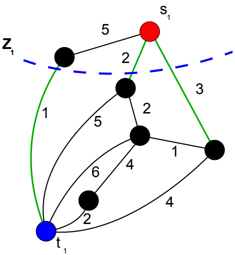

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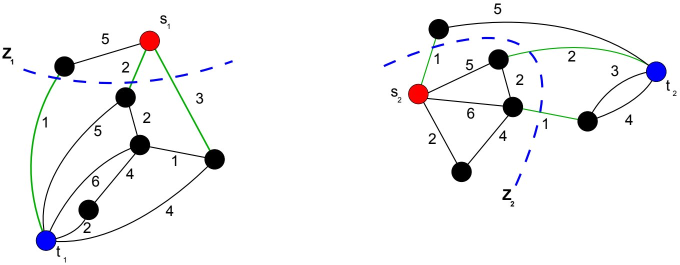

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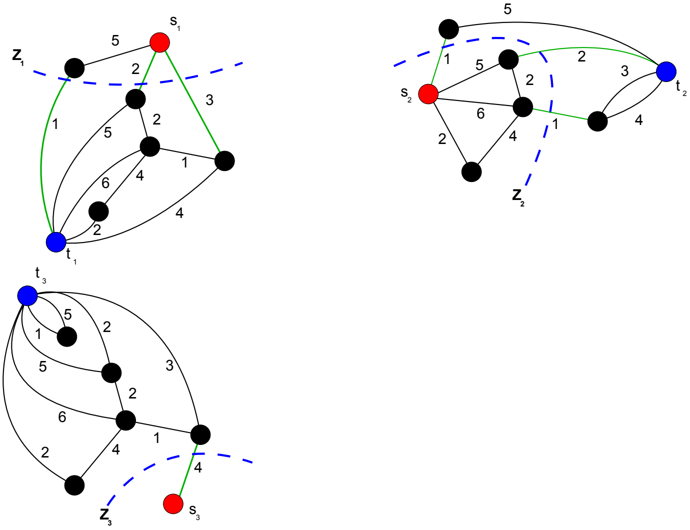

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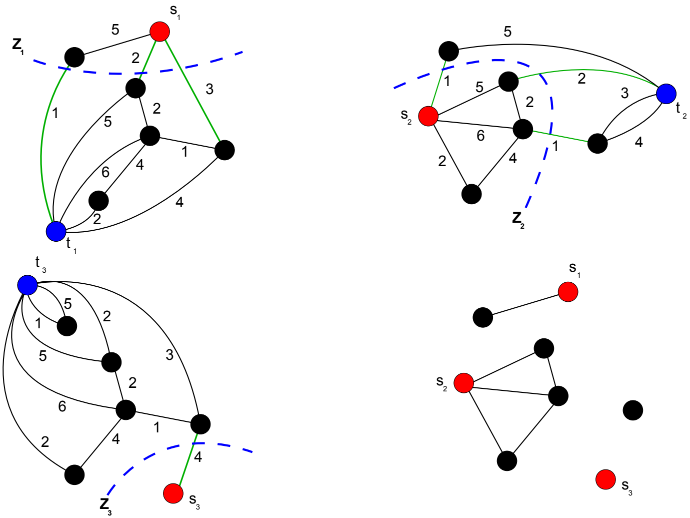
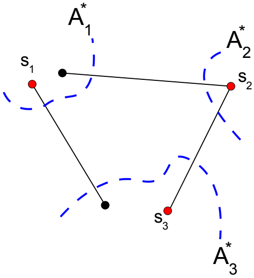


Cost = 11

Analysis

- A^*: optimal solution
- A_i^*: edges in A^* with exactly one endpoint in the partition containing s_i

$$w(Z_1) \leq w(A_1^*)$$
$$w(Z_2) \leq w(A_2^*)$$
$$\vdots$$
$$w(Z_k) \leq w(A_k^*)$$



$$w(\bigcup_i Z_i) \leq \sum_{i=1}^{k} w(A_i^*) = 2 \cdot OPT$$

$$w(\bigcup_{i=1}^{k-1} Z_i) \leq (1 - \tfrac{1}{k}) w(\bigcup_i Z_i) \leq 2(1 - \tfrac{1}{k}) \cdot OPT$$

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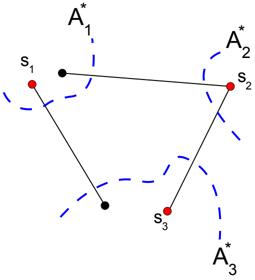

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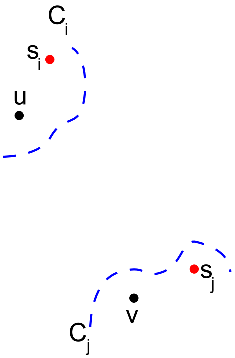
Theorem

The above gives a $(2 - \frac{2}{k})$*-approximation for multiway cut.*

Outline

LP Relaxation

- W.l.o.g., any feasible solution will partition the vertices into $C_1, C_2, \cdots, C_k$
- $s_i \in C_i$
- $\delta(C_i)$: set of edges leaving C_i
- $v_i = \begin{cases} 1, & \text{if } v \in C_i \\ 0, & \text{otherwise} \end{cases}$
- $z_e^i = \begin{cases} 1, & \text{if } e \in \delta(C_i) \\ 0, & \text{otherwise} \end{cases}$
- Obj: $\min \bigcup_i w(\delta(C_i))$
- In other words:
  $\min \frac{1}{2} \sum_{e \in E} w_e \sum_{i=1}^{k} z_e^i$



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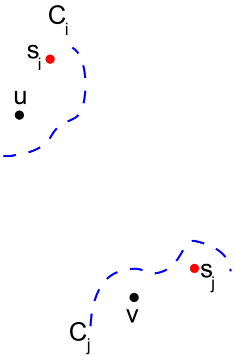

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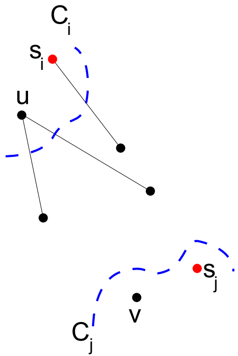

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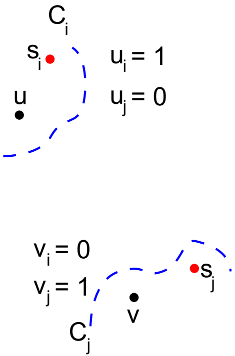

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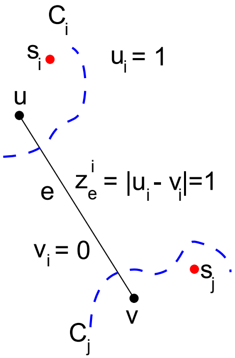

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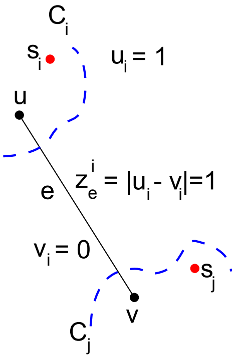

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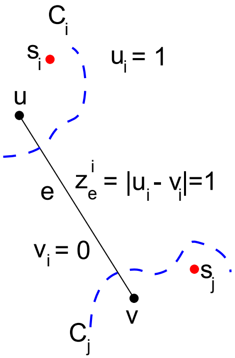

Constructing the LP

Integer Program

$$\text{minimize} \quad \frac{1}{2}\sum_{e\in E} w_e \sum_{i=1}^{k} z_e^i \tag{1}$$

$$\text{s.t.} \quad \sum_{i=1}^{k} v_i = 1, \qquad \forall v \in V \tag{2}$$

$$z_e^i = |u_i - v_i|, \qquad \forall e = (u,v) \in E, \quad 1 \le i \le k \tag{3}$$

$$s_{i_i} = 1, \qquad \forall s \in S \tag{4}$$

$$v_i \in \{0,1\}, \qquad \forall v \in V, \quad 1 \le i \le k \tag{5}$$

Constructing the LP

LP Relaxation

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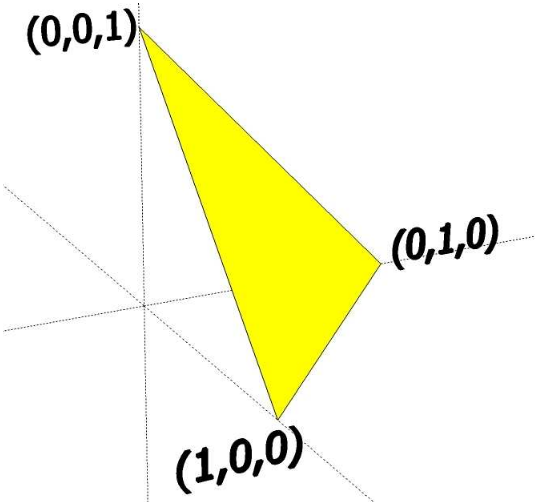
$$s_{i_i} = 1, \qquad \forall s \in S \tag{4}$$

$$v_i \ge 0, \qquad \forall v \in V, \quad 1 \le i \le k \tag{5}$$

Interpreting the LP

- $x_v = (x_v^1, x_v^2, ..., x_v^k)$, a point in $\mathbb{R}^k$
- $x_v^i \leftarrow v_i$
- all coordinates of x_v sum to 1
- x_v belongs to the $(k-1)$-simplex
- x_{s_i}: unit vector with i^{th} coordinate equal to 1
- Let $d_e = \frac{1}{2}\sum_{i=1}^{k} z_e^i = \frac{1}{2}\sum_{i=1}^{k} |x_u^i - x_v^i|$

Δ_3 Simplex



Interpreting the LP

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Constructing the LP

LP Relaxation

minimize $\frac{1}{2} \sum_{e \in E} w_e \sum_{i=1}^{k} z_e^i$

s.t.

$$\sum_{i=1}^{k} v_i = 1, \qquad \forall v \in V$$

$$z_e^i = |u_i - v_i|, \qquad \forall e = (u,v) \in E, \quad 1 \leq i \leq k$$

$$s_{i_i} = 1, \qquad \forall s \in S$$

$$v_i \geq 0, \qquad \forall v \in V, \quad 1 \leq i \leq k$$

LP Relaxation

LP

$$\text{minimize} \quad \sum_{e \in E} w_e d_e$$

$$\text{s.t.} \quad \sum_{i=1}^{k} x_v^i = 1, \qquad \forall v \in V$$

$$d_e = \frac{1}{2} \sum_{i=1}^{k} |x_u^i - x_v^i| \qquad \forall e = (u, v) \in E$$

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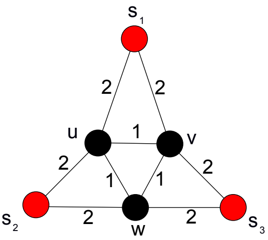
LP Relaxation

Is this a valid LP relaxation?



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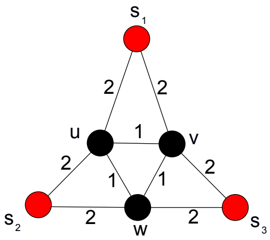

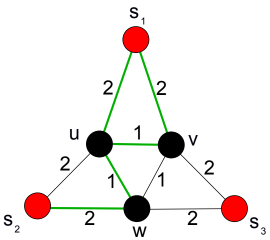

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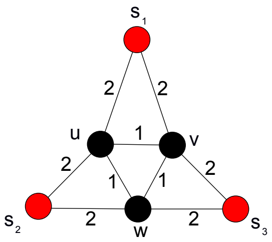

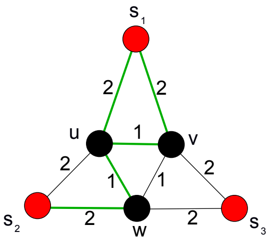

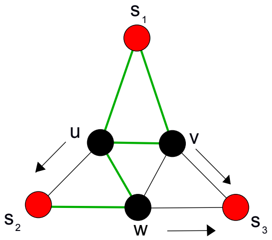

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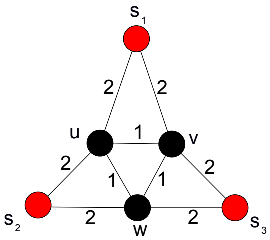

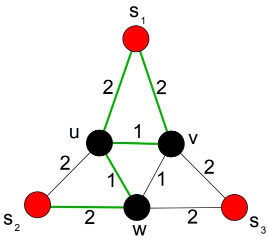

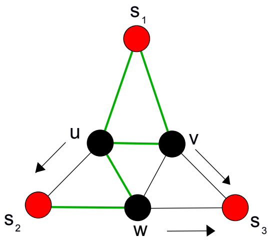

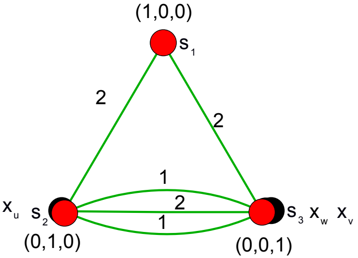

LP Relaxation

LP

minimize $\sum_{e \in E} w_e d_e$

s.t.

$$\sum_{i=1}^{k} x_v^i = 1, \qquad \forall v \in V$$

$$d_e = \frac{1}{2} \sum_{i=1}^{k} |x_u^i - x_v^i| \qquad \forall e = (u, v) \in E$$

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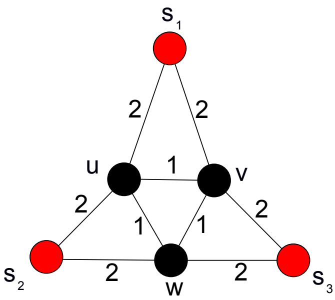
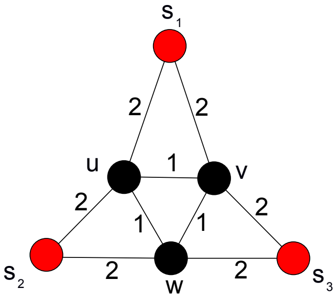
Outline

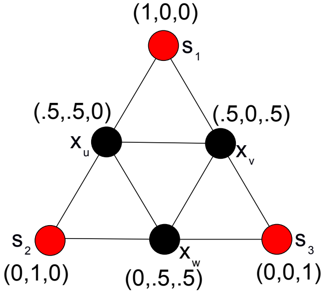
LP Solution: Example



OPT = 8

LP Solution: Example





OPT = 8

LP-OPT = 7.5

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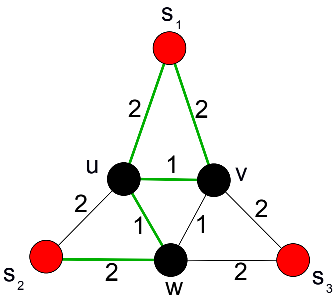

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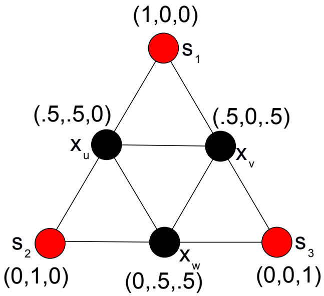

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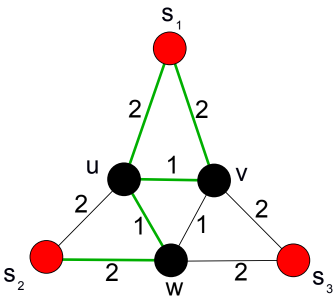

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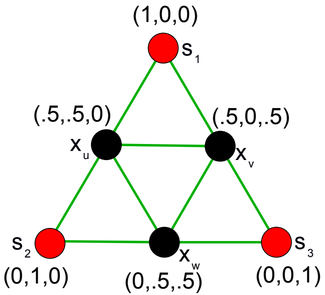


LP-OPT = 7.5

Outline

Notation

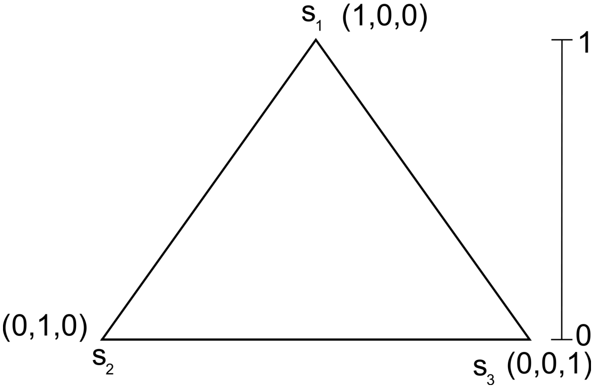
- Assume that the LP gives a solution in which, for any edge (u,v), its endpoints differ in at most two coordinates.
- E_i: edges whose endpoints differ in coordinate i.
- W_i: $\sum_{e \in E_i} w_e d_e$
- $B(s_i, \rho)$: $\{v \in V | x_v^i \geq \rho\}$.
- C_i: partition containing s_i

Algorithm

Algorithm

1. Compute optimum LP solution.
2. Renumber the terminals so that $W_1 \leq \cdots \leq W_k$
3. Pick uniformly at random $\rho \in (0,1)$ and $\sigma \in \{(1,2,\dots,k-1,k),(k-1,k-2,\dots,1,k)\}$.
4. **for** $i = 1$ to $k-1$:
   $$C_{\sigma(i)} \leftarrow B(s_i,\rho) - \bigcup_{j<i} C_{\sigma(j)}$$
5. $C_k \leftarrow$ all remaining vertices
6. C: set of edges that run between sets $C_1,\dots,C_k$
7. **return** C

Example of Algorithm Partitions

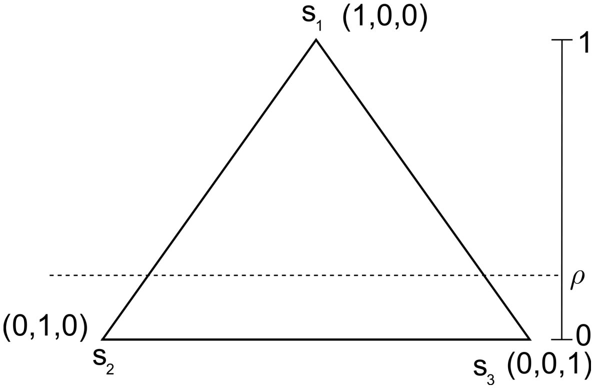


$$\rho < \frac{1}{2},\ \sigma = (1,2,3)$$

Example of Algorithm Partitions

Multiway Cut

Sean Lowen



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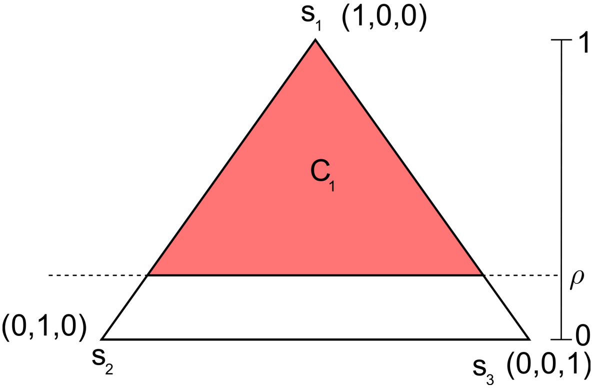

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Example of Algorithm Partitions

Multiway Cut

Sean Lowen

2-2/k Approx

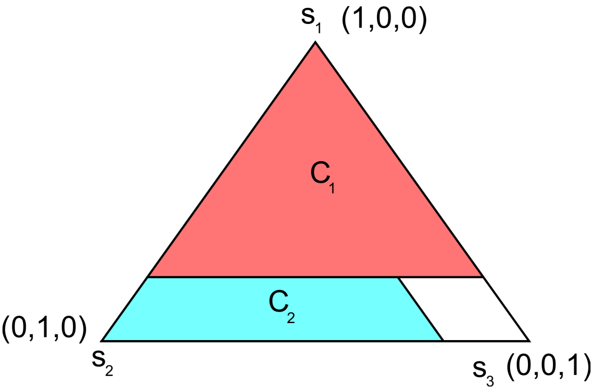
Simplex

Understanding LP

Randomization

Analysis

Modifying LP Solution



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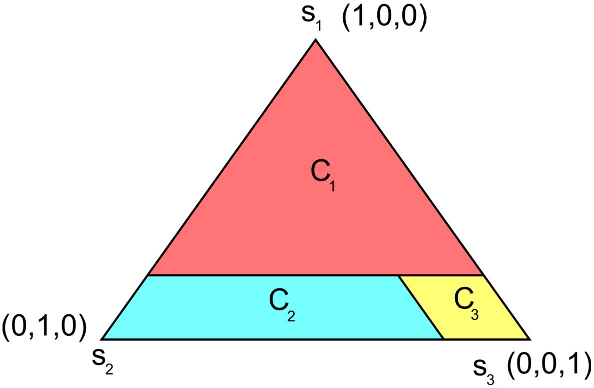

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Outline

Analysis

Lemma

If $e \in E - E_k$, $\mathbf{Pr}[e \in C] \leq 1.5d_e$
If $e \in E_k$, $\mathbf{Pr}[e \in C] \leq d_e$

- Consider edge $e = (u, v)$, where u and v differ in coordinates i and j
- By definition, $x_u^i - x_v^i = x_v^j - x_u^j = d_e$
- Edge e will only be cut if x_u and x_v end up in different partitions, i.e., ρ falls in either the (x_u^i, x_v^i) or (x_v^j, x_u^j) intervals.
- Let's call those intervals α and β, respectively.

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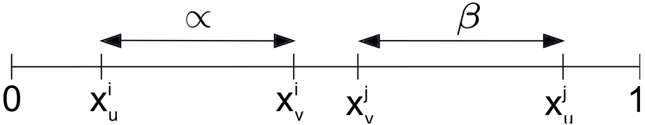
Analysis: $e \in E \setminus E_k$

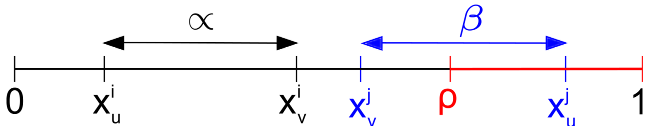
Edge $e = (u, v)$ differs in coordinates i & j



$$\mathbf{Pr}[e \in C] = \mathbf{Pr}[\rho \in (\alpha \cup \beta)] = |\alpha| + |\beta| \leq 2d_e$$

Analysis: $e \in E \setminus E_k$

Case I: $\rho \in \beta$ and $\sigma = (\ldots, j, \ldots, i, \ldots)$



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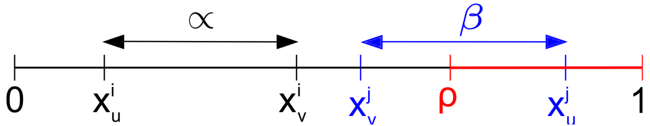
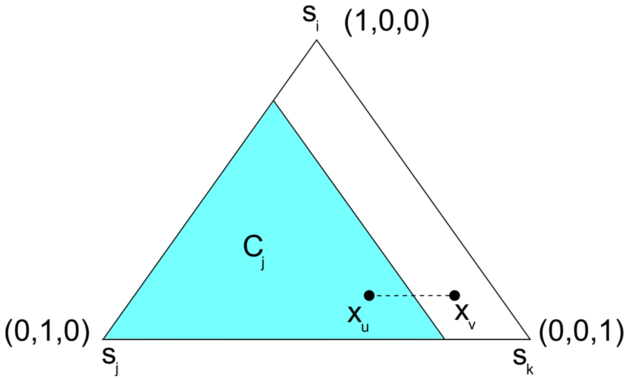
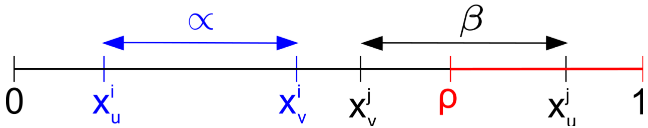




Case II: $\rho \in \beta$ and $\sigma = (\ldots, i, \ldots, j, \ldots)$



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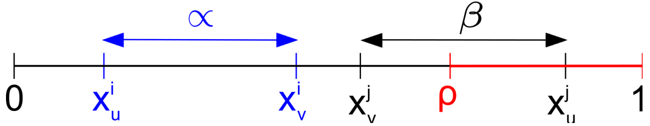

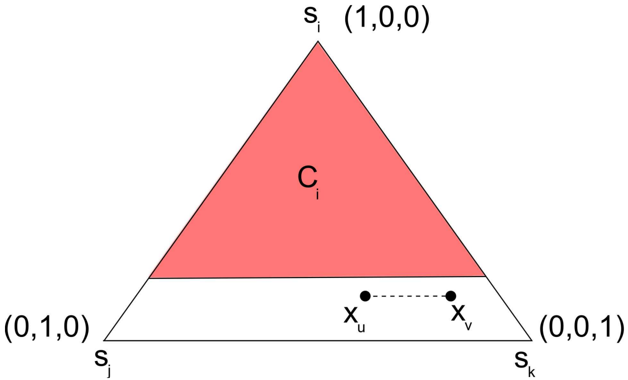

Analysis: $e \in E \setminus E_k$

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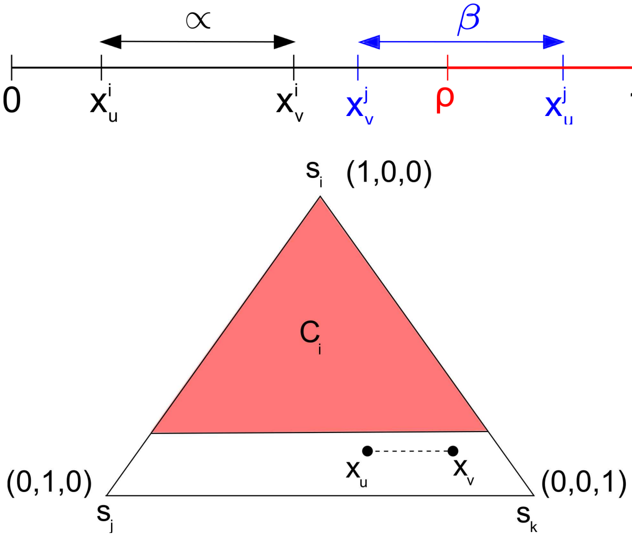

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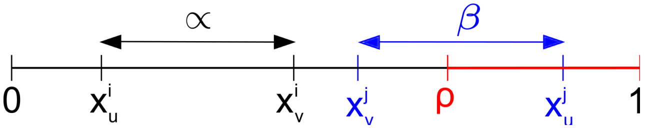
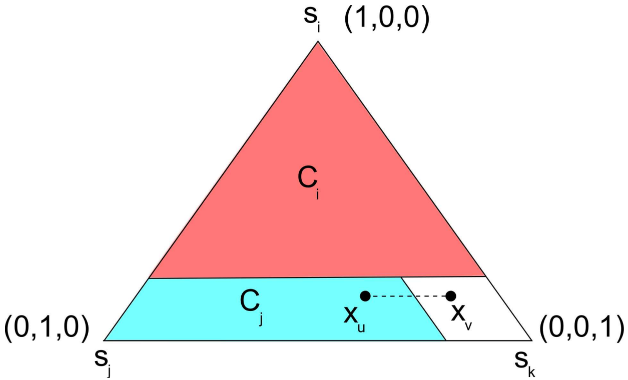
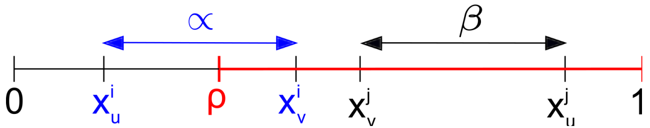




Case III: $\rho \in \alpha$ and $\sigma = (\ldots, i, \ldots, j, \ldots)$



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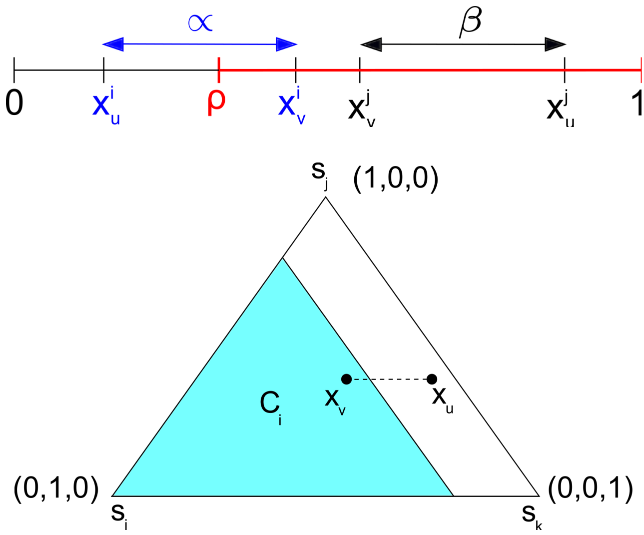
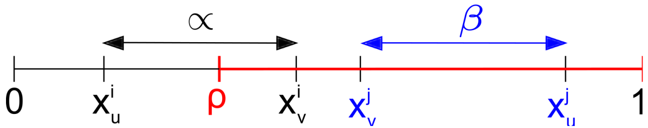
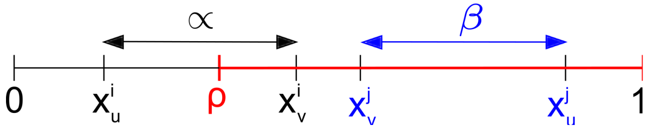
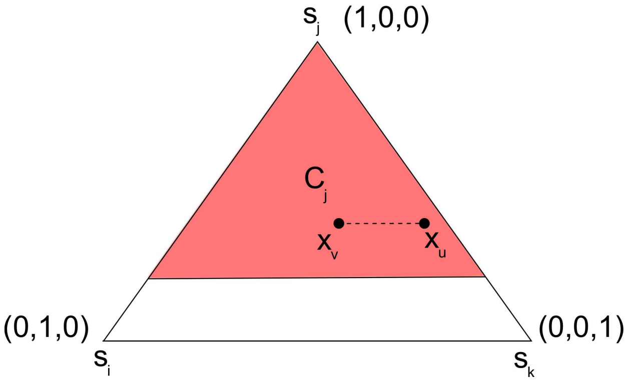


Case IV: $\rho \in \alpha$ and $\sigma = (\ldots, j, \ldots, i, \ldots)$



Case IV: $\rho \in \alpha$ and $\sigma = (\dots, j, \dots, i, \dots)$





Analysis: $e \in E \setminus E_k$

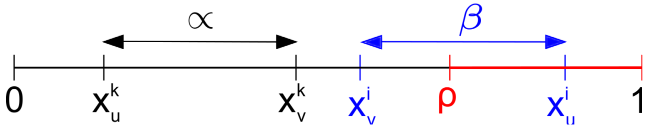
$$\begin{aligned}\mathbf{Pr}[e \in C] &= \mathbf{Pr}[\rho \in \beta] + \mathbf{Pr}[(\rho \in \alpha) \wedge (\sigma = (\ldots, i, \ldots, j, \ldots))] \\ &= |\beta| + \frac{|\alpha|}{2} \leq 1.5 d_e\end{aligned}$$

Lemma

If $e \in E - E_k$, $\mathbf{Pr}[e \in C] \leq 1.5 d_e$

Analysis: $e \in E_k$

An edge in E_k can only be cut if $\rho \in \beta$.



- σ *always* $= (\ldots, i, \ldots, k)$
- k gets all remaining vertices
- $\mathbf{Pr}[e \in C] = |\beta| = d_e$

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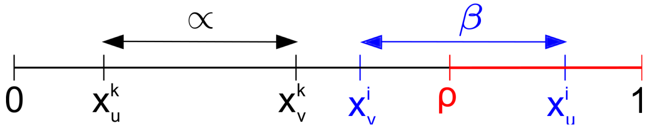

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Lemma

If $e \in E_k, \mathbf{Pr}[e \in C] \leq d_e$

Approximation

$$OPT_f = \sum_{e \in E} w_e d_e$$

Note that every edge belongs to exactly 2 of the E_i sets.

$$\sum_{i=1}^{k} W_i = 2 \cdot OPT_f$$

Since W_k is the largest of the sets:

$$W_k = \sum_{e \in E_k} w_e d_e \geq \frac{2}{k} \cdot OPT_f$$

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Approximation

$$
\begin{aligned}
\mathbf{E}[w(C)] &= \sum_{e \in E} w_e \mathbf{Pr}[e \in C] \\
&= \sum_{e \in E - E_k} w_e \mathbf{Pr}[e \in C] + \sum_{e \in E_k} w_e \mathbf{Pr}[e \in E_k] \\
&\leq 1.5 \sum_{e \in E - E_k} w_e d_e + \sum_{e \in E_k} w_e d_e \\
&= 1.5 \sum_{e \in E} w_e d_e - 0.5 \sum_{e \in E_k} w_e d_e \\
&= 1.5 \sum_{e \in E} w_e d_e - 0.5 W_k \\
&\leq (1.5 - 1/k) \cdot OPT_f
\end{aligned}
$$

Conclusion

Theorem

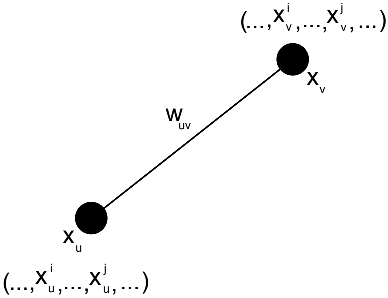
There is a $(1.5 - \frac{1}{k})$*-approximation for multiway cut.*

Outline

Modifying the LP

Convert the LP solution to another LP solution of same cost in which each edge differs in at most two coordinates.

- x_u & x_v differ in more than two coordinates.
- i: coordinate in which x_u and x_v differ the least. Let $x_v^i > x_u^i$.
- Let $\alpha = x_v^i - x_u^i$
- There is a coordinate j s.t. $x_u^j - x_v^j \geq \alpha$
- $\forall k \neq i, j$, $x_w^k \leftarrow x_v^k$
- $w_{uw}, w_{wv} \leftarrow w_{uv}$



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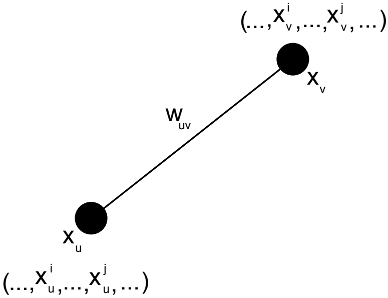

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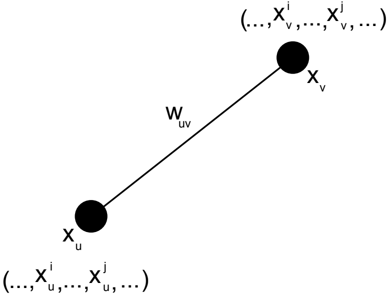

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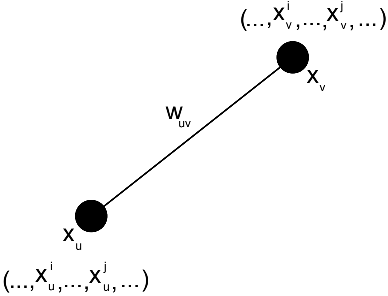

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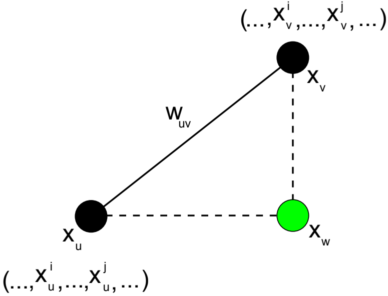

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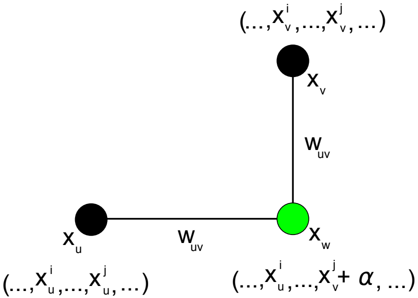

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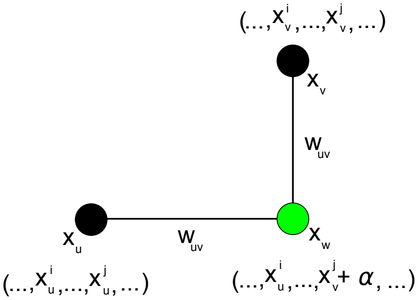
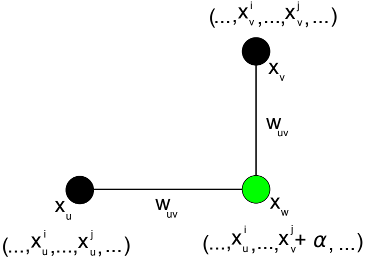


Does $w_{uv}d_{uv} = w_{uw}d_{uw} + w_{wv}d_{wv}$?

$$w_{uv}d_{uv} \stackrel{?}{=} w_{uv}d_{uw} + w_{uv}d_{wv}$$
$$d_{uv} \stackrel{?}{=} d_{uw} + d_{wv}$$

Let $d_{uv} = \sigma$

$$d_{uw} = \sigma - 2\alpha$$
$$d_{wv} = 2\alpha$$
$$d_{uw} + d_{wv} = 2\alpha + \sigma - 2\alpha$$
$$d_{uw} + d_{wv} = \sigma = d_{uv}$$



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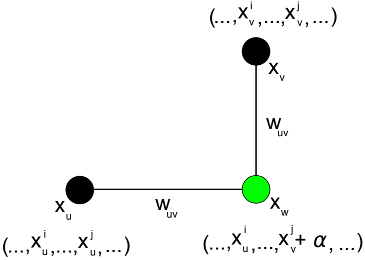

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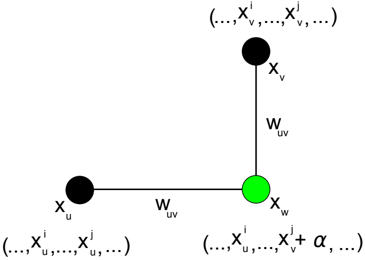

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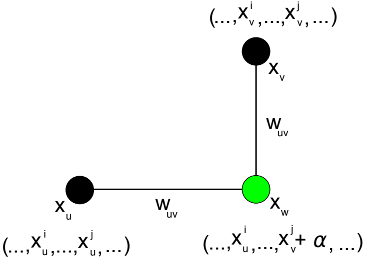

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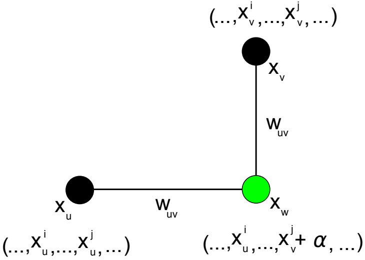
$$d_{uw} + d_{wv} = 2\alpha + \sigma - 2\alpha$$

$$d_{uw} + d_{wv} = \sigma = d_{uv}$$



Does $w_{uv} d_{uv} = w_{uw} d_{uw} + w_{wv} d_{wv}$?

$$w_{uv} d_{uv} \stackrel{?}{=} w_{uv} d_{uw} + w_{uv} d_{wv}$$

$$d_{uv} \stackrel{?}{=} d_{uw} + d_{wv}$$

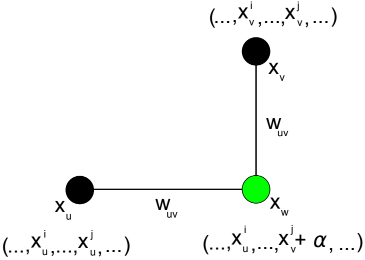
$$\text{Let } d_{uv} = \sigma$$

$$d_{uw} = \sigma - 2\alpha$$

$$d_{wv} = 2\alpha$$

$$d_{uw} + d_{wv} = 2\alpha + \sigma - 2\alpha$$

$$d_{uw} + d_{wv} = \sigma = d_{uv}$$



Does $w_{uv} d_{uv} = w_{uw} d_{uw} + w_{wv} d_{wv}$?

$$w_{uv} d_{uv} \stackrel{?}{=} w_{uv} d_{uw} + w_{uv} d_{wv}$$

$$d_{uv} \stackrel{?}{=} d_{uw} + d_{wv}$$

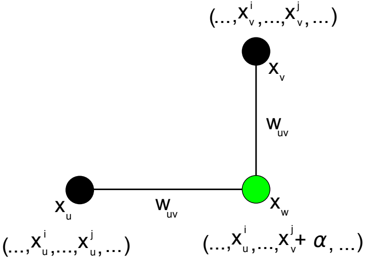
Let $d_{uv} = \sigma$

$$d_{uw} = \sigma - 2\alpha$$

$$d_{wv} = 2\alpha$$

$$d_{uw} + d_{wv} = 2\alpha + \sigma - 2\alpha$$

$$d_{uw} + d_{wv} = \sigma = d_{uv}$$



Lemma

There is a way to modify the LP s.t. for any edge $e = (u, v)$, x_u and x_v differ in 0 or 2 coordinates, at no additional cost.

References

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End

Thank you!