Capacitated Vertex Cover

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The Problem

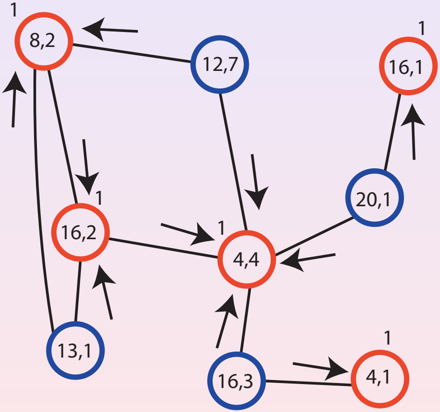
Input:

- Graph $G = (V, E)$
- For each vertex, v:
  - A weight, w_v, assigned to each copy of v
  - A capacity, k_v, assigned to each copy of v.

Objective:

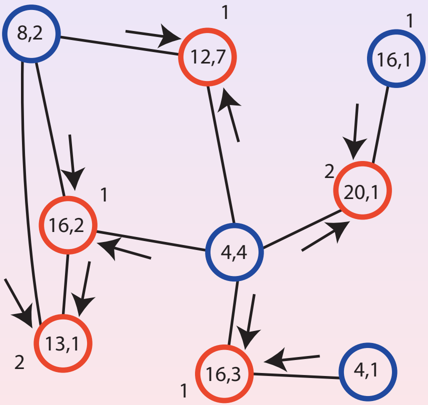
- Find a multiset of vertices, C, of minimum weight s.t. no vertex copy is assigned more than k_v edges.

The Problem



$$\text{Cost} = 8 \cdot 1 + 16 \cdot 1 + 4 \cdot 1 + 4 \cdot 1 + 16 \cdot 1 = 48$$

The Problem



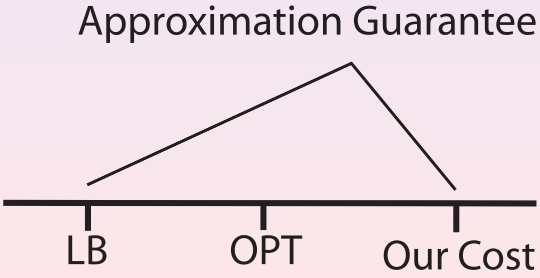
Cost = $12 \cdot 1 + 16 \cdot 1 + 13 \cdot 2 + 16 \cdot 1 + 20 \cdot 2 = 110$

Approximation Algorithms

- CVC is NP-hard.
  - No optimal poly-time algorithms are known
- β-approximation alg. for a minimization problem P
  - Poly-time algorithm.
  - For every instance, I, of P, the alg. produces solution of cost at most $\beta \cdot OPT(I)$
  - We don't know $OPT(I)$...

Approximation Algorithms

- Compute a lower bound on *OPT*.
- Compare cost of our solution with the lower bound.



IP Formulation

- x_v: the number of copies of v in C.
- γ_{ev}: given the value 1 if edge e is assigned to v, 0 o.w.

$$
\begin{aligned}
\min \quad & \sum_{v \in V} w_v \cdot x_v \\
& \gamma_{eu} + \gamma_{ev} \geq 1, \quad \forall e = (u, v) \\
& k_v \cdot x_v - \sum_{e \in E(v)} \gamma_{ev} \geq 0, \quad \forall v \\
& x_v \in \mathbb{N} \\
& \gamma_{ev} \in \{0, 1\}
\end{aligned}
$$

The Relaxation

$$\min \quad \sum_{v \in V} w_v \cdot x_v$$

$$\gamma_{eu} + \gamma_{ev} \geq 1, \quad \forall e = (u, v)$$

$$k_v \cdot x_v - \sum_{e \in (v)} \gamma_{ev} \geq 0, \quad \forall v$$

$$\boxed{x_v \geq 0}$$

$$\boxed{\gamma_{ev} \geq 0}$$

Why Do This?

- LP can be solved in polynomial time
- Every solution to the IP is also a solution to the LP
- Hence:

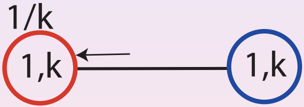
$$\boxed{OPT_{LP} \leq OPT_{IP}}$$

- We can use OPT_{LP} as a lower bound on OPT_{IP}

Integrality Gap

$$\gamma_{eu} + \gamma_{ev} \geq 1$$

$$k_v \cdot x_v - \sum_{e \in E(v)} \gamma_{ev} \geq 0$$

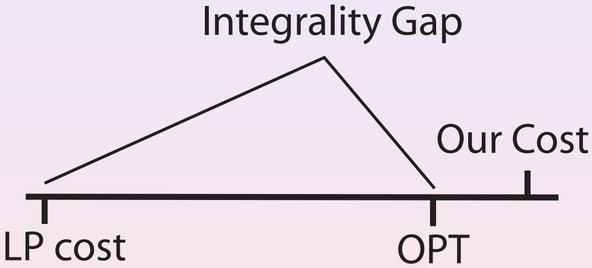


IP Cost = 1

LP Cost = $1/k$

- $\frac{1}{k}$ can be arbitrarily small: there is an unbounded gap.
- OPT_{LP} is a poor lower bound.

Integrality Gap



- β, our appx. guarantee, would be unboundedly large.

New IP

- Add a new constraint: $x_v \geq \gamma_{ev}$; the IP still represents CVC.

$$
\begin{aligned}
\min \quad & \sum_{v \in V} w_v \cdot x_v \\
& \gamma_{eu} + \gamma_{ev} \geq 1, \quad \forall e = (u, v) \\
& k_v \cdot x_v - \sum_{e \in E(v)} \gamma_{ev} \geq 0, \quad \forall v \\
& \boxed{x_v \geq \gamma_{ev}, \quad \forall v \forall e \in E(v)} \\
& x_v \in \mathbb{N} \\
& \gamma_{ev} \in \{0, 1\}
\end{aligned}
$$

New LP

$$\min \quad \sum_{v \in V} w_v \cdot x_v$$

$$\gamma_{eu} + \gamma_{ev} \geq 1, \quad \forall e = (u, v)$$

$$k_v \cdot x_v - \sum_{e \in E(v)} \gamma_{ev} \geq 0, \quad \forall v$$

$$x_v \geq \gamma_{ev}, \quad \forall v \forall e \in E(v)$$

$$\boxed{x_v \geq 0}$$

$$\boxed{\gamma_{ev} \geq 0}$$

The Integrality Gap Fixed

$$\gamma_{eu} + \gamma_{ev} \geq 1$$

$$k_v \cdot x_v - \sum_{e \in E(v)} \gamma_{ev} \geq 0$$

$$x_v \geq \gamma_{ev}$$

New IP Cost = 1

New LP

New LP Cost = 1
We must choose a vertex once.

A Rounding Algorithm

- Solve the LP.
- Set $\widehat{\gamma_{ev}} = 1$ whenever $\gamma^*_{ev} \geq \frac{1}{2}$ and set $\widehat{\gamma_{ev}} = 0$ otherwise.
- Set $\widehat{x_v} = \lceil \frac{\text{number of edges assigned to v}}{k_v} \rceil = \lceil \frac{\sum_{e \in E(v)} \widehat{\gamma_{ev}}}{k_v} \rceil$.

Feasibility

$$
\begin{aligned}
\min \quad & \sum_{v \in V} w_v \cdot x_v \\
& \gamma_{eu} + \gamma_{ev} \geq 1, \quad \forall e = (u, v) \\
& k_v \cdot x_v - \sum_{e \in E(v)} \gamma_{ev} \geq 0, \quad \forall v \\
& x_v \geq \gamma_{ev}, \quad \forall v \forall e \in E(v) \\
& x_v \in \mathbb{N} \\
& \gamma_{ev} \in \{0, 1\}
\end{aligned}
$$

Feasibility

1. $\gamma_{eu} + \gamma_{ev} \geq 1, \quad \forall e = (u, v)$
   - $\gamma^*_{eu} \geq \frac{1}{2}$ or $\gamma^*_{ev} \geq \frac{1}{2} \rightarrow \widehat{\gamma_{eu}} = 1$ or $\widehat{\gamma_{ev}} = 1$.
2. $k_v \cdot x_v - \sum_{e \in E(v)} \gamma_{ev} \geq 0, \quad \forall v$
   - $\widehat{x_v} = \lceil \frac{\text{number of edges assigned to v}}{k_v} \rceil = \lceil \frac{\sum_{e \in E(v)} \widehat{\gamma_{ev}}}{k_v} \rceil \geq \frac{\sum_{e \in E(v)} \widehat{\gamma_{ev}}}{k_v}$ forces this to be true.
3. $x_v \geq \gamma_{ev}, \quad \forall v \forall e \in E(v)$
   - $\widehat{x_v} = \lceil \frac{\text{number of edges assigned to v}}{k_v} \rceil \geq 1$ whenever $\widehat{\gamma_{ev}} = 1$ for some e hitting v.

Analysis

1. $k_v \cdot x_v^* \geq \sum_{e \in E(v)} \gamma_{ev}^*$
2. $\widehat{\gamma_{ev}} \leq 2 \cdot \gamma_{ev}^*$
3. If $\widehat{x_v} \neq 0$ then $x_v^* \geq \frac{1}{2}$.

$$\begin{aligned} \widehat{x_v} \neq 0 &\rightarrow \widehat{\gamma_{ev}} = 1 \\ &\rightarrow \gamma_{ev}^* \geq \frac{1}{2} \\ &\rightarrow x_v^* \geq \frac{1}{2} \qquad (x_v^* \geq \gamma_{ev}^*) \end{aligned}$$

Analysis

$$\text{Our Cost} = \sum_{v \in V} w_v \cdot \widehat{x_v} = \sum_{v | \widehat{x_v} \neq 0} w_v \cdot \widehat{x_v}$$

$$= \sum_{v | \widehat{x_v} \neq 0} w_v \cdot \left\lceil \frac{\sum_{e \in E(v)} \widehat{\gamma_{ev}}}{k_v} \right\rceil$$

$$\leq \sum_{v | \widehat{x_v} \neq 0} w_v \cdot \left\lceil \frac{\sum_{e \in E(v)} 2 \cdot \gamma^*_{ev}}{k_v} \right\rceil$$

$$\leq \sum_{v | \widehat{x_v} \neq 0} w_v \cdot \left\lceil \frac{2 \cdot (x^*_v k_v)}{k_v} \right\rceil = \sum_{v | \widehat{x_v} \neq 0} w_v \cdot \lceil 2 x^*_v \rceil$$

$$\leq 4 \cdot \sum_{v | \widehat{x_v} \neq 0} w_v \cdot x^*_v$$

Primal-Dual Method

$$\boxed{Dual_{Feasible} \leq Dual_{OPT} = Primal_{OPT} \leq Primal_{Feasible}}$$

- Construct the dual LP
- Construct an algorithm that manually tightens dual constraints to obtain a “maximal” dual solution

The Primal

$$\min \quad \sum_{v \in V} w_v \cdot x_v$$

$$\gamma_{eu} + \gamma_{ev} \geq 1, \quad \forall e = (u, v) \quad \boxed{\alpha_e}$$

$$k_v \cdot x_v - \sum_{e \in E(v)} \gamma_{ev} \geq 0, \quad \forall v \quad \boxed{q_v}$$

$$x_v \geq \gamma_{ev}, \quad \forall v \forall e \in E(v) \quad \boxed{l_{ev}}$$

$$x_v \geq 0$$

$$\gamma_{ev} \geq 0$$

The Dual

$$\max \quad \sum_{e \in E} \alpha_e$$

$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v, \quad \forall v$$

$$\alpha_e \leq q_v + l_{ev}, \quad \forall v \forall e \in E(v)$$

$$q_v \geq 0$$

$$l_{ev} \geq 0$$

$$\alpha_e \geq 0$$

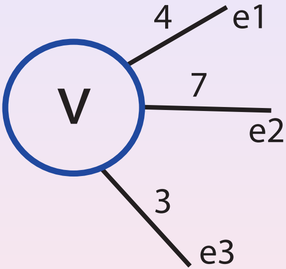
The Dual

- The dual to the CVC problem is a packing problem.
- α_e: the weight packed into edge e.
- q_v: v's global ability to absorb edge weight.
- l_{ev}: v's ability to absorb edge weight from edge e.

The Dual

$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v = 17$$

$$\alpha_{e_1} = 4 \leq q_v + l_{e_1 v}$$
$$\alpha_{e_2} = 7 \leq q_v + l_{e_2 v}$$
$$\alpha_{e_3} = 3 \leq q_v + l_{e_3 v}$$



1. Set $l_{e_1 v} = 4$, $l_{e_2 v} = 7$, $l_{e_3 v} = 3$.
2. Set $q_v = 7$.
3. Set $l_{e_1 v} = 1$, $l_{e_2 v} = 4$, $l_{e_3 v} = 0$, $q_v = 3$.

Primal-Dual Algorithm

- Maximize $\sum_{e \in E} \alpha_e$ by raising all α_e simultaneously *while making sure the constraints are satisfied*.
- Put v in our cover when
$$k_v q_v + \sum_{e \in \delta(v)} l_{ev} \leq w_v$$
becomes tight.
- Delete v and $\delta(v)$ from the graph.
- Repeat until all edges are removed from the graph.

Primal-Dual Algorithm

$$\max \quad \sum_{e \in E} \alpha_e$$

$$k_v q_v + \sum_{e \in \delta(v)} l_{ev} \leq w_v, \quad \forall v$$

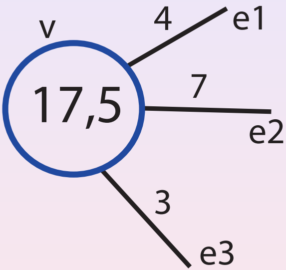
$$\boxed{\alpha_e \leq q_v + l_{ev}, \quad \forall v \forall e \in \delta(v)}$$

- As we raise α_e we need to raise either q_v or l_{ev}. Which?
- We want $k_v q_v + \sum l_{ev}$ to approach w_v slowly.
- Lower $k_v q_v + \sum l_{ev}$ values let us raise α_e values higher.

Raising Variables

$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v = 17$$

$$\alpha_{e_1} = 4 \leq q_v + l_{e_1 v}$$
$$\alpha_{e_2} = 7 \leq q_v + l_{e_2 v}$$
$$\alpha_{e_3} = 3 \leq q_v + l_{e_3 v}$$

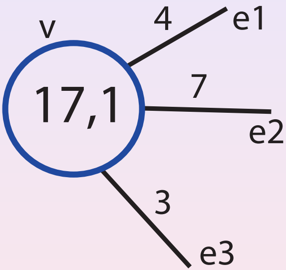


- $l_{e_1 v} = 4$, $l_{e_2 v} = 7$, $l_{e_3 v} = 3$
- $k_v q_v + \sum l_{ev} = 14$.
- $q_v = 7$
- $k_v q_v + \sum l_{ev} = 35$.

Raising Variables

$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v = 17$$

$$\alpha_{e_1} = 4 \leq q_v + l_{e_1 v}$$
$$\alpha_{e_2} = 7 \leq q_v + l_{e_2 v}$$
$$\alpha_{e_3} = 3 \leq q_v + l_{e_3 v}$$



- $l_{e_1 v} = 4$, $l_{e_2 v} = 7$, $l_{e_3 v} = 3$
- $k_v q_v + \sum l_{ev} = 14.$
- $q_v = 7$
- $k_v q_v + \sum l_{ev} = 7.$

Low and High Degree Vertices

- A low degree vertex is one where $|\delta(v)| \leq k_v$.
  - Raise l_{ev} for all edges in $\delta(v)$.
- A high degree vertex is one where $|\delta(v)| > k_v$.
  - Raise q_v.

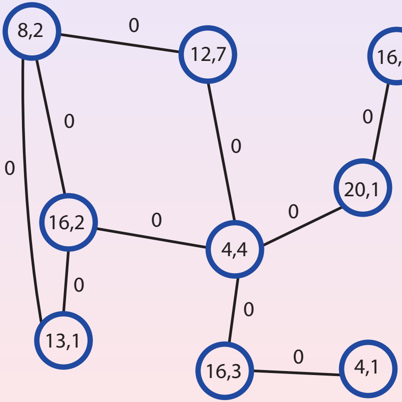
Primal-Dual Algorithm

- Maximize $\sum_{e \in E} \alpha_e$ by raising all α_e simultaneously while making sure the constraints are satisfied.
- Put v in our cover when

$$k_v q_v + \sum_{e \in \delta(v)} l_{ev} \leq w_v$$

  becomes tight.
- Delete v and $\delta(v)$ from the graph.
- Repeat until all edges are removed from the graph.

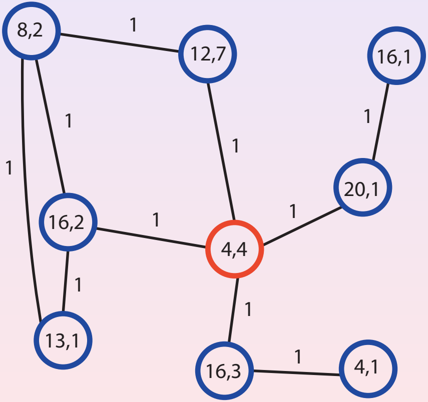
The Algorithm



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

$$\alpha_e \leq q_v + l_{ev}$$

The Algorithm



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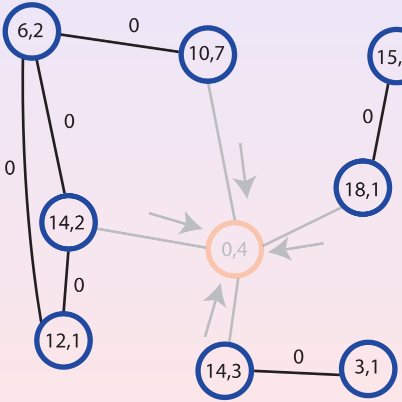
The Algorithm



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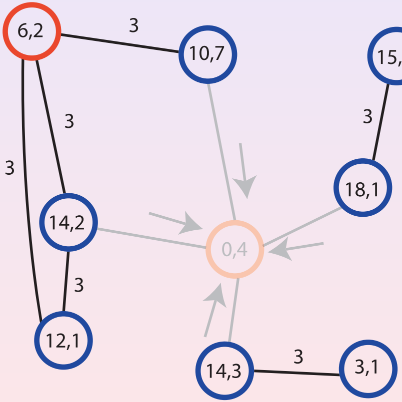
The Algorithm



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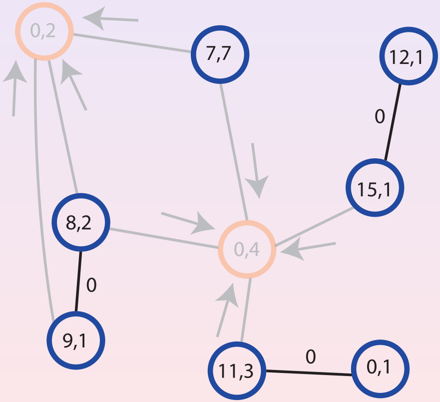
The Algorithm



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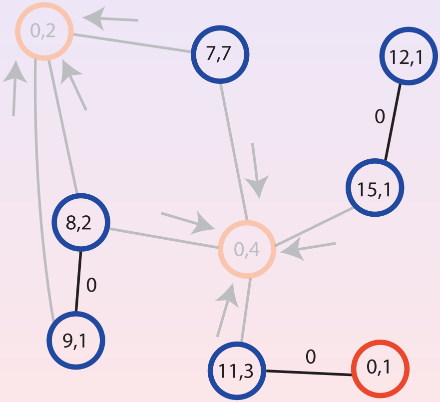
The Algorithm



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

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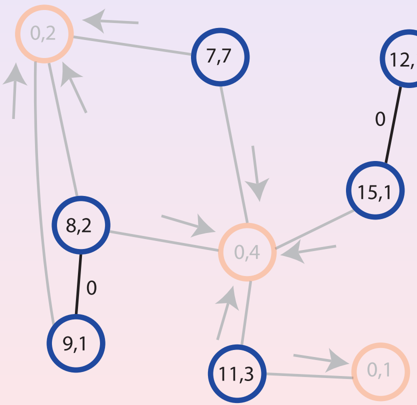
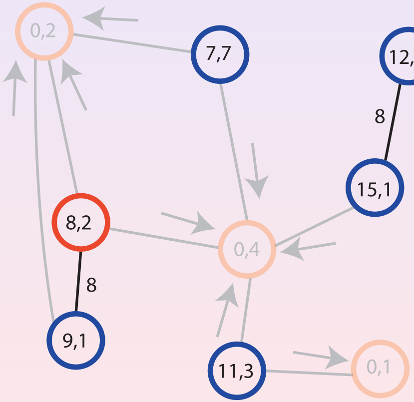
The Algorithm



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \le w_v$$

$$\alpha_e \le q_v + l_{ev}$$

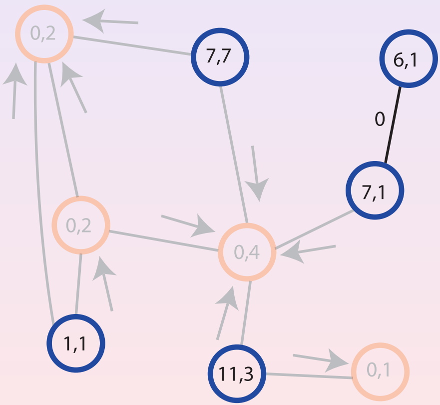
The Algorithm



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

$$\alpha_e \leq q_v + l_{ev}$$

The Algorithm



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

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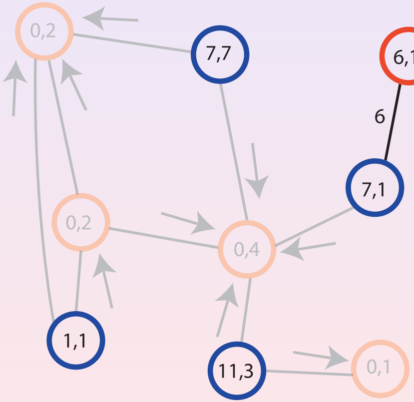
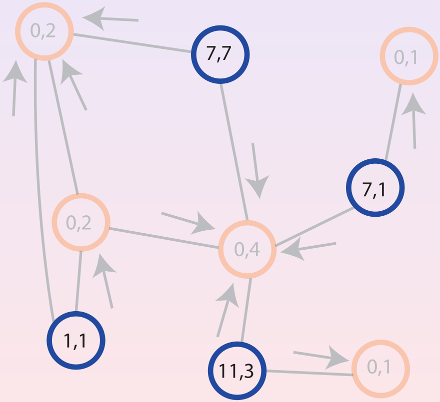
The Algorithm



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

$$\alpha_e \leq q_v + l_{ev}$$

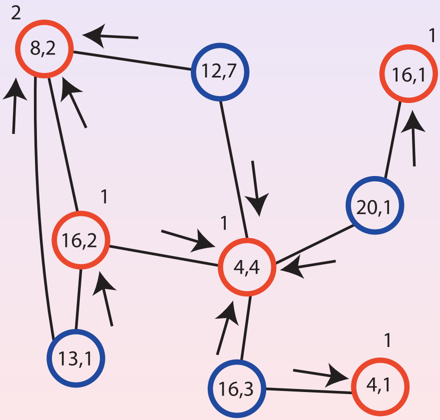
The Algorithm



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

$$\alpha_e \leq q_v + l_{ev}$$

The Algorithm: Cost = 56



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

$$\alpha_e \leq q_v + l_{ev}$$

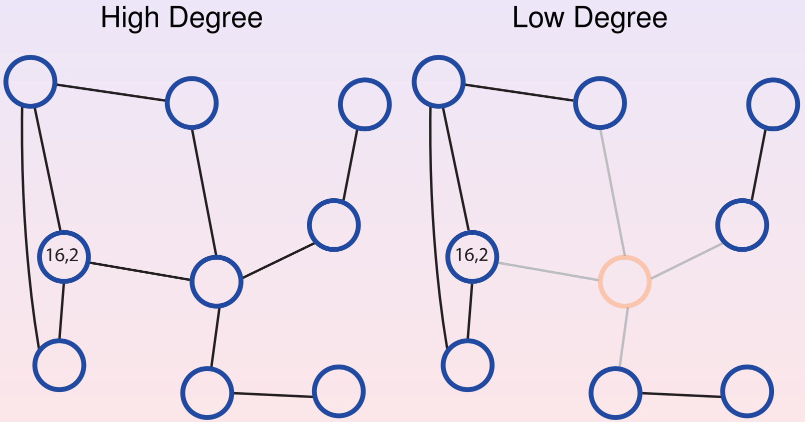
Analysis

- We want to relate our cost to the dual feasible solution.
- When v is put in our cover
  - $k_v q_v + \sum_{e \in \delta(v)} l_{ev} = w_v$
  - $q_v + l_{ev} = \alpha_e$
- We've raised q_v and l_{ev} variables differently for different vertices.

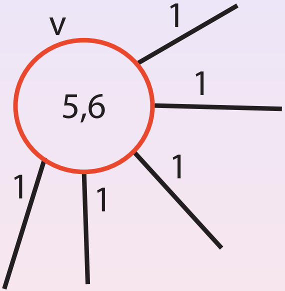
3 Types of Vertices

- A vertex is always low-degree: “LD” type vertices.
- A vertex is always high-degree: “HD” type vertices.
- A vertex changes from high-degree to low-degree: “T” type vertices.

High-Degree to Low-Degree Vertex



Low-Degree Vertices



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

$$\alpha_e \leq q_v + l_{ev}$$

- We need 1 copy.
- $l_{ev} = 1$
- $q_v = 0$

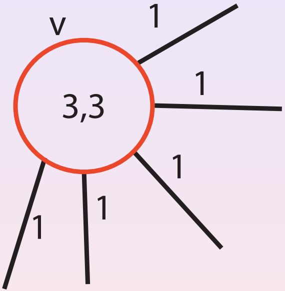
Low-Degree Vertices

- We always need 1 copy.
- $\alpha_e = l_{ev}$
- $\sum l_{ev} = w_v$
- We pay $w_v = \boxed{\sum \alpha_e}$

$$\boxed{k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v}$$

$$\boxed{\alpha_e \leq q_v + l_{ev}}$$

High-Degree Vertices



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

$$\alpha_e \leq q_v + l_{ev}$$

- We need 2 copies.
- $l_{ev} = 0$
- $q_v = 1$

High-Degree Vertices

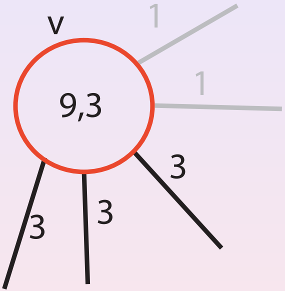
- We need $\lceil \frac{|\delta(v)|}{k_v} \rceil$ copies.
- $\lceil \frac{|\delta(v)|}{k_v} \rceil \leq 2 \cdot \frac{|\delta(v)|}{k_v}$
- $\alpha_e = q_v$
- $k_v \cdot q_v = w_v$

$$\boxed{k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v}$$

$$\boxed{\alpha_e \leq q_v + l_{ev}}$$

- We pay $\lceil \frac{|\delta(v)|}{k_v} \rceil w_v \leq 2 \frac{|\delta(v)|}{k_v} w_v = 2 \frac{|\delta(v)|}{k_v} k_v q_v = \boxed{2 \sum \alpha_e}$

Transitional Vertices when $|\delta(v)| = k_v$



$$k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v$$

$$\alpha_e \leq q_v + l_{ev}$$

- We need 1 copy.
- $l_{ev} = 2$
- $q_v = 1$

Transitional Vertices when $|\delta(v)| = k_v$

- We always need 1 copy.
- $|\delta(v)| = k_v$
- $\alpha_e = l_{ev} + q_v$
- $k_v q_v + \sum l_{ev} = w_v$

$$\boxed{k_v q_v + \sum_{e \in E(v)} l_{ev} \leq w_v}$$

$$\boxed{\alpha_e \leq q_v + l_{ev}}$$

- We pay $w_v = k_v q_v + \sum l_{ev} = k_v(q_v + l_{ev}) = \boxed{\sum \alpha_e}$

Transitional Vertices: The General Case

To generalize we need more notation:

- Let $D(v)$ be the edges incident on v at the time v went from a HD vertex to a LD vertex.
- $|D(v)| = k_v$

Transitional Vertices: The General Case

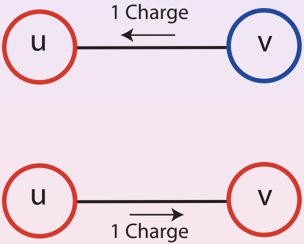
$$\text{We pay } w_v \cdot 1 = q_v k_v + \sum_{e \in \delta(v)} l_{ev}$$
$$\leq q_v k_v + \sum_{e \in D(v)} l_{ev}$$
$$= \sum_{e \in D(v)} l_{ev} + q_v$$
$$= \boxed{\sum_{e \in D(v)} \alpha_e}$$

Transitional Vertices

- Every edge in $D(v)$, not $\delta(v)$, is charged.
- Edges in $D(v) - \delta(v)$ are charged from both ends.

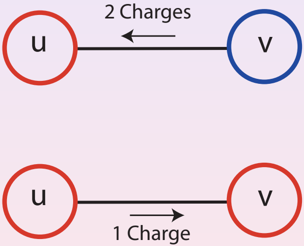
Charging Edges

Type 1: u is LD, v is T



(u, v) is charged $2\alpha_e$.

Type 2: u is HD, v is T



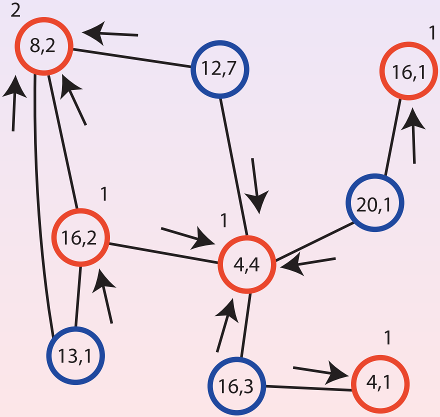
(u, v) is charged $3\alpha_e$.

Charging Edges

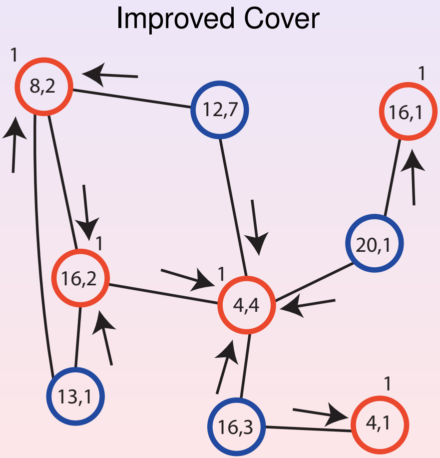
- Each edge is charged at most $3\alpha_e$
- Our Cost $\leq 3\sum_e \alpha_e$.
- We have a 3 approximation.

Improvement

Our Cover



Improvement



References

S. Guha, R. Hassin, S. Khuller, and E. Or. Capacitated Vertex Covering with Applications, Symposium on Discrete Algorithms (SODA) 2002.

Thank You!