Iterative Rounding and Relaxation

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Iterative Rounding and Relaxation

Ingredients:

- Linear Program
- Theorem about individual variable values in LP solution

Technique:

- Solve LP
- Round some variables
- Remove variables, relax constraints
- Iterate

Brief History

Survivable Network Design

- Jain (1998)

MBDST

- Goemans (2006)
- Singh and Lau (2007, 2008)
- Bansal, Khandekar, Nagarajan (2008)

Outline

Vertex Cover

Input:

- A graph $G = (V, E)$
- Non-negative costs on vertices c_v

Output:

- A minimum-cost collection of vertices so that each edge in G is incident on at least one vertex in the collection

Vertex Cover

$$
\begin{aligned}
\min \sum_{v \in V} c_v x_v & \\
x_u + x_v \geq 1 & \qquad \forall e = (u, v) \\
x_v \geq 0 & \qquad \forall v \in V
\end{aligned}
$$

Vertex Cover: Main Theorem

Theorem (Nemhauser-Trotter)

In a basic optimal LP solution, each $x_v \in \{\frac{1}{2}, 1, 0\}$

Simple 2-appx algorithm:

- Solve the Vertex Cover LP
- Include all vertices with $x_v \neq 0$ in our cover

MBDST: Problem Statement

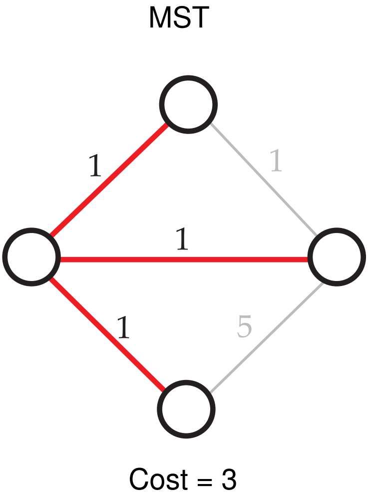
Input:

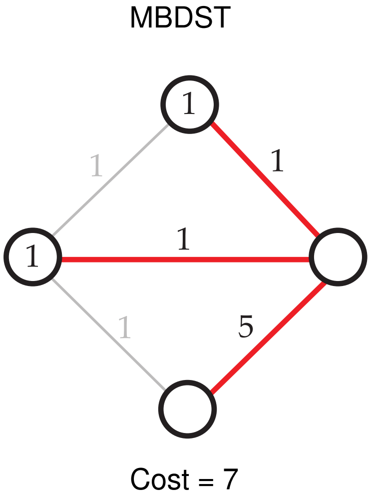
- A graph $G = (V, E)$
- Costs $c_e \geq 0$ for all $e \in E$
- A set $W \subseteq V$
- Degree bounds b_v for all $v \in W$

Output:

- Find a min-cost spanning tree (V, F) that doesn't violate degree bounds.

Example





Outline

MBDST Properties

Notation:

- S: any subset of vertices
- $E(S)$: edges with both endpoints in S
- F: set of edges in MBDST

Properties:

Spanning: Exactly $|V| - 1$ edges in F

Acyclic: For $|S| \geq 2$, at most $|S| - 1$ edges of F in $E(S)$

Degree Bounds: At most b_v edges of F incident on v

Integer Program

- $x_e = 1$ if $e \in F$ and $x_e = 0$ otherwise

$$\min \sum_{e \in E} c_e x_e \qquad \text{(Objective)}$$

$$\sum_{e \in E} x_e = |V| - 1 \qquad (1)$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \qquad \forall S \subset V, |S| \geq 2 \qquad (2)$$

$$\sum_{e \in \delta(v)} x_e \leq b_v \qquad \forall v \in W \qquad (3)$$

$$x_e \in \{0, 1\} \qquad \forall e \in E$$

Linear Program

- $x_e = 1$ if $e \in F$ and $x_e = 0$ otherwise

$$\min \sum_{e \in E} c_e x_e \qquad \text{(Objective)}$$

$$\sum_{e \in E} x_e = |V| - 1 \qquad (1)$$

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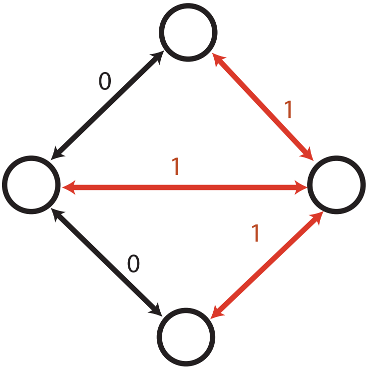
$$x_e \geq 0 \qquad \forall e \in E$$

LP Properties

- There are exponentially many constraints (2)
- Ellipsoid method
- Separation oracle
  - (1) and (3) are easy to check
  - (2) requires work

Skip Oracle

Separation Oracle: Flow Network



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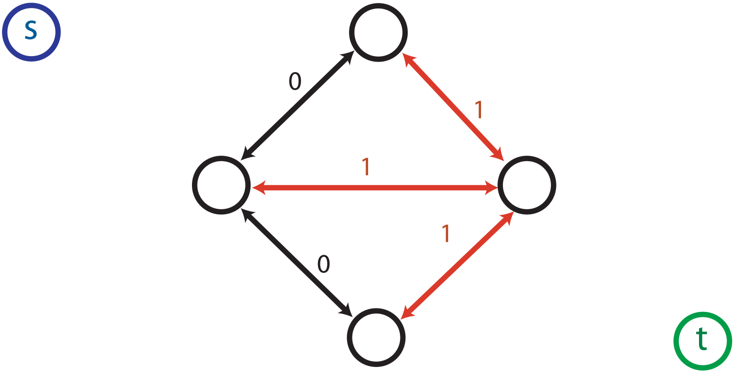

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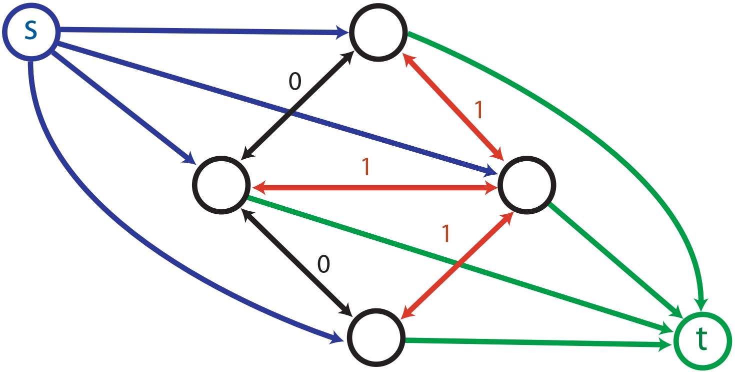

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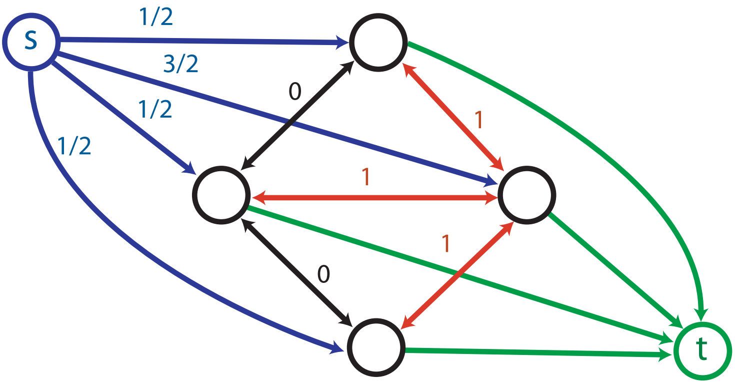

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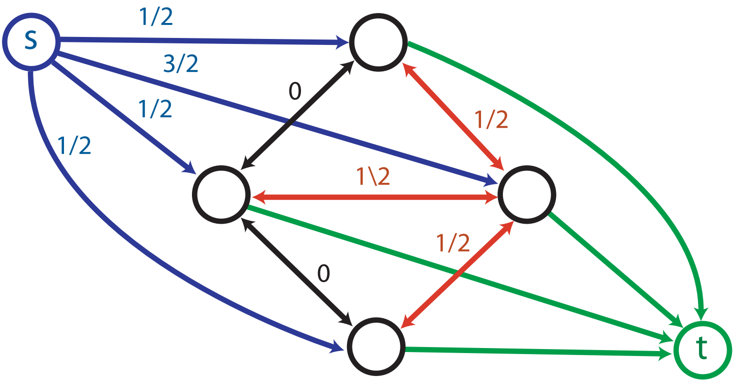

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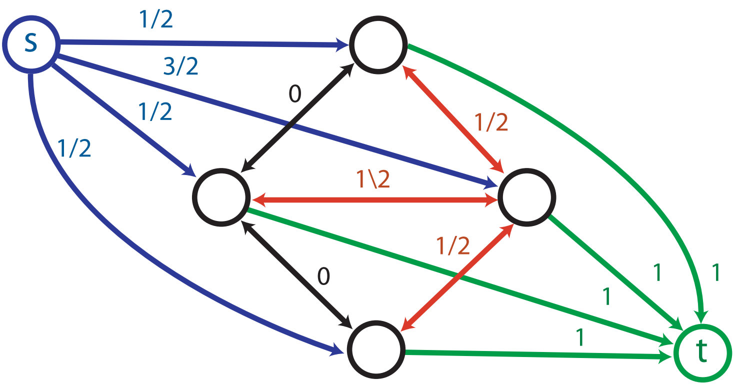
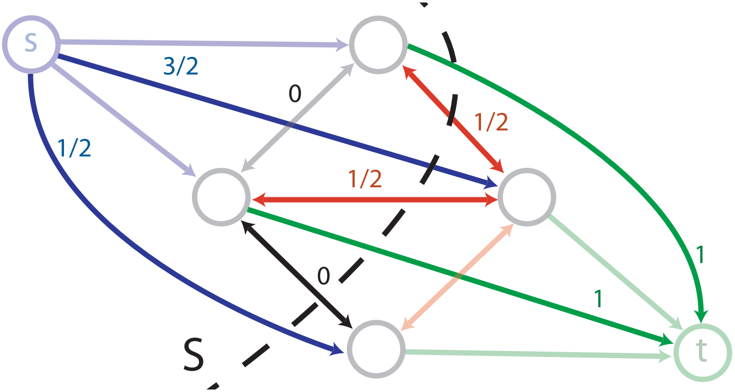


Separation Oracle: *s*-*t* cut



$$\text{Capacity} = 2 + \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 4$$

Separation Oracle

- The capacity across S is $|V| + (|S| - 1) - \sum_{e \in E(S)} x_e$
- The capacity across S is at least $|V|$ iff $\sum_{e \in E(S)} x_e \leq |S| - 1$
- The max-flow from s to t is $|V|$ iff (2) are satisfied

Outline

Main Theorem

- $\bar{x} =< x_1, x_2, \ldots, x_{|E|} >$: solution to LP
- *Support*$(\bar{x})$: set of edges s.t. $x_e > 0$

Theorem

For any basic solution $\bar{x}$ to the linear program either:

1. - $\exists v$ *with exactly one incident edge* $e \in$ *Support*$(\bar{x})$
   - $x_e = 1$
2. $\exists v \in W$ *with at most 3 edges of Support*$(\bar{x})$ *incident on* v

- Condition 1 identifies a leaf in the tree
- Condition 2 identifies a vertex with sufficiently small number of nonzero incident edges

Algorithm

$F = \emptyset$
While $|V| > 1$

- $\bar{x} \leftarrow$ LP solution on $< G, W >$
- Remove all edges e with $x_e = 0$
- If condition 1 is satisfied by $\bar{x}$
    - Add (u, v) to F
    - Remove v and (u, v) from G
    - If $u \in W$ reduce b_u by 1
- If condition 2 is satisfied by $\bar{x}$
    - Remove v from W

Linear Program

$$\min \sum_{e \in E} c_e x_e \qquad \text{(Objective)}$$

$$\sum_{e \in E} x_e = |V| - 1 \qquad (1)$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \qquad \forall S \subset V, |S| \geq 2 \qquad (2)$$

$$\sum_{e \in \delta(v)} x_e \leq b_v \qquad \forall v \in W \qquad (3)$$

$$x_e \geq 0 \qquad \forall e \in E$$

LP Relationships

In each iteration LP is in the same *family*

- Same separation oracle
- The Main Theorem applies to each LP

If condition 1 is satisfied LP is incrementally modified:

- Delete an x_e variable
- Modify (1) constraint
- Remove some (2) constraints
- Modify some (3) constraints

If condition 2 is satisfied LP is incrementally modified:

- Remove a (3) constraint

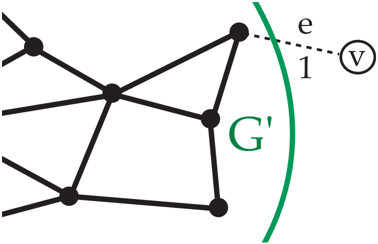
Outline

Outline

Bounding Cost

Theorem

The tree returned by our algorithm has cost at most LP OPT



LP: current lin. prog.
LP$'$: new lin. prog.
F': MBDST in G'

IH: $cost(F') \leq \mathrm{LP}'(G')$

$$\begin{aligned} cost(F') + c_e &\leq \mathrm{LP}'(G') + c_e x_e \\ &\leq \mathrm{LP}(G') + c_e x_e \\ &= \mathrm{LP}(G) \end{aligned}$$

Bounding Cost

Lemma

$LP(G')$ is a feasible solution to $LP'(G')$

Changes:

1. -1 on RHS, -1 on LHS
2. Remove constraints
3. -1 on RHS, -1 on LHS; Remove constraints

$$\sum_{e \in E} x_e = |V| - 1 \qquad (1)$$

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Min-Cost Spanning Trees

Recap:

- Spanning tree has optimal cost
- Degree bounds?

Implications:

Theorem

For any basic solution $\bar{x}$ to the linear program either:

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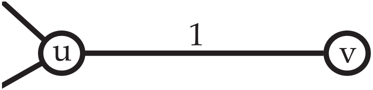
- When $W = \emptyset$ we have OPT

Outline

Degree Bounds



- $b_v, b_u \geq 1$
- b_v never violated
- b_u "adjusted"

Algorithm

$\bar{x} \leftarrow$ LP solution on $< G, W >$
Remove $e \notin Support(\bar{x})$
If condition 1 is satisfied by $\bar{x}$

- **Add (u, v) to F**
- **Remove v and (u, v) from G**
- **If $u \in W$ reduce b_u by 1**

If condition 2 is satisfied by $\bar{x}$

- Remove v from W

Degree Bounds

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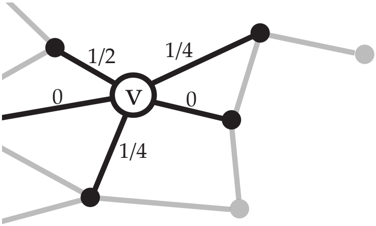

- $b_v \geq 1$
- All 3 edges may be in F
- b_v violated by at most 2

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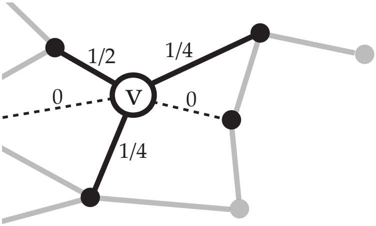
Degree Bounds



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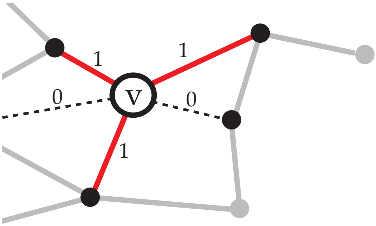
Degree Bounds



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- If $u \in W$ reduce b_u by 1

If condition 2 is satisfied by $\bar{x}$

- **Remove v from W**

Analysis Summary

Cost:

- Tree has cost no more than *OPT*

Degree Bounds:

- No degree bound violated by more than 2

Theorem (Goemans)

The algorithm for MBDST produces a spanning tree in which the degree of v is at most $b_v + 2$ for $v \in W$ and has cost no greater than OPT

Outline

Main Theorem

Theorem

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$$x_e \geq 0 \qquad \forall e \in E$$

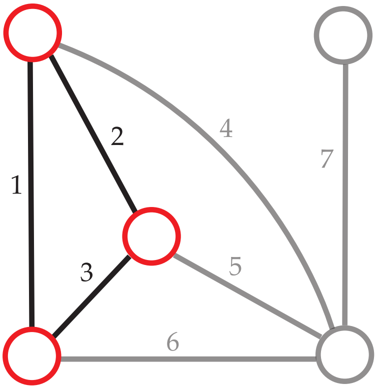
Laminar Lemma

Lemma

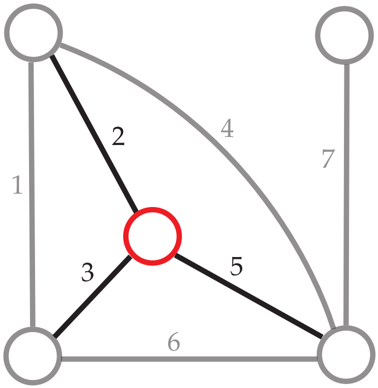
For any basic LP solution $\bar{x}$ there is a $Z \subseteq W$ and a collection $\mathcal{L}$ of $S \subseteq V$ where:

1. $\forall S \in \mathcal{L}$, *S is tight;* $\forall v \in Z$, *v is tight*
2. *The vectors* $\chi_{E(S)}$ *and* $\chi_{\delta(v)}$ *are independent*
3. $|\mathcal{L}| + |Z| = |Support(\bar{x})|$
4. $\mathcal{L}$ *is laminar*

Characteristic Vector



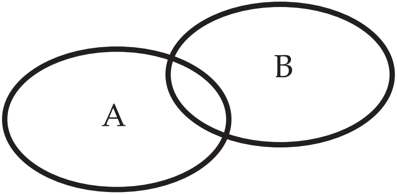
$\chi_{E(S)} = < 1, 1, 1, 0, 0, 0, 0 >$



$\chi_{\delta(v)} = < 0, 1, 1, 0, 1, 0, 0 >$

Laminar Sets

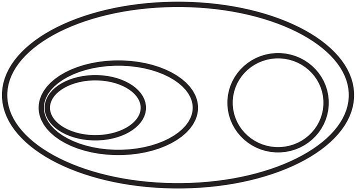
Intersecting Sets



- $A \cap B \neq \emptyset$
- $A - B \neq \emptyset$
- $B - A \neq \emptyset$

Laminar Sets

- No intersecting sets



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Property of $\mathcal{L}$

Lemma

If all $S \in \mathcal{L}$ contain at least 2 vertices then $|\mathcal{L}| \leq |V| - 1$

- Use induction on $|V|$
- Base case: $|V| = 2$
- Induction step
  - Shrink smallest set to vertex
  - Generates $\mathcal{L}'$ and V'
  - $\mathcal{L}'$ is laminar
  - $|\mathcal{L}'| = |\mathcal{L}| - 1$
  - $|V'| \leq |V| - 1$
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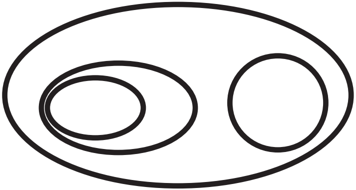

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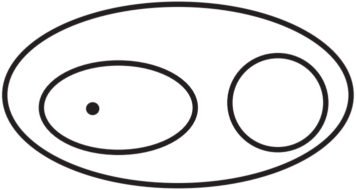

Property of $Support(\bar{x})$

Lemma

$$|Support(\bar{x})| < |V| + |W|$$

Recall property 3 of Laminar Lemma: $|\mathcal{L}| + |Z| = |Support(\bar{x})|$

$$\begin{aligned} |Support(\bar{x})| &= |\mathcal{L}| + |Z| \\ &\leq |\mathcal{L}| + |W| \\ &< |V| + |W| \qquad \text{(Previous Lemma)} \end{aligned}$$

From Laminar Lemma to Main Theorem

Theorem

For any basic solution $\bar{x}$ to the linear program either:

1. - $\exists v$ *with exactly one incident edge* $e \in Support(\bar{x})$
   - $x_e = 1$
2. $\exists v \in W$ *with at most 3 edges of* $Support(\bar{x})$ *incident on* v

Suppose Main Theorem wasn't true:

- For every $v \in V$ there are at least 2 edges incident on it
- For every $v \in W$ there are at least 4 edges incident on it

$$|Support(\bar{x})| \geq \frac{1}{2}(2(|V| - |W|) + 4(|W|))$$
$$= |V| + |W| \qquad \text{(Contradiction!)}$$

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- $x_e \leq 1$
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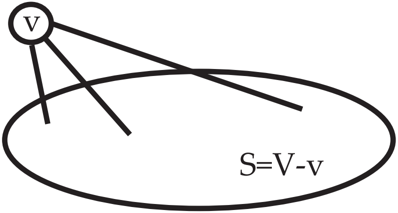

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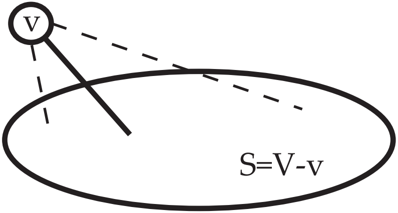

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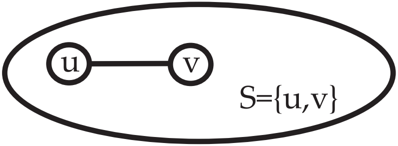

- $\sum_{e \in E(S)} x_e \leq |V| - 2$
- $\sum_{e \in E} x_e = |V| - 1$
- $\sum_{e \in \delta(v)} x_e \geq 1$
- $x_e \geq 1$
- $x_e \leq 1$
- $x_e = 1$

Outline

Laminar Lemma

Lemma

For any basic LP solution $\bar{x}$ there is a $Z \subseteq W$ and a collection $\mathcal{L}$ of $S \subseteq V$ where:

1. $\forall S \in \mathcal{L}$, *S is tight;* $\forall v \in Z$, *v is tight*
2. *The vectors* $\chi_{E(S)}$ *and* $\chi_{\delta(v)}$ *are independent*
3. $|\mathcal{L}| + |Z| = |Support(\bar{x})|$
4. $\mathcal{L}$ *is laminar*

LP Background

$$\min \sum c_i x_i \qquad \text{(Objective)}$$
$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \geq b_1 \qquad (1)$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \geq b_2 \qquad (2)$$
$$... = ...$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \geq b_m \qquad \text{(m)}$$
$$x_i \geq 0 \qquad \text{(Non-Negative)}$$

LP Background

- Constraints define half-spaces
- Objective is a hyperplane
- **Solution always a corner**
- $\geq n$ tight constraints
- Constraints lin. ind.

Linear Program

$$\min \sum c_i x_i$$

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \geq b_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \geq b_2 \quad (2)$$

$$... = ...$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \geq b_m \quad (\text{m})$$

$$x_i \geq 0$$

LP Background

- Constraints define half-spaces
- Objective is a hyperplane
- **Solution always a corner**
- $\geq |E|$ tight constraints
- Constraints lin. ind.

MBDST LP

$$\min \sum_{e \in E} c_e x_e$$

$$\sum_{e \in E} x_e = |V| - 1 \quad (1)$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \quad (2)$$

$$\sum_{e \in \delta(v)} x_e \leq b_v \quad (3)$$

$$x_e \geq 0$$

Laminar Lemma

Lemma

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1. $\forall S \in \mathcal{L}$, *S is tight;* $\forall v \in Z$, *v is tight*
2. *The vectors* $\chi_{E(S)}$ *and* $\chi_{\delta(v)}$ *are independent*
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Laminar Lemma

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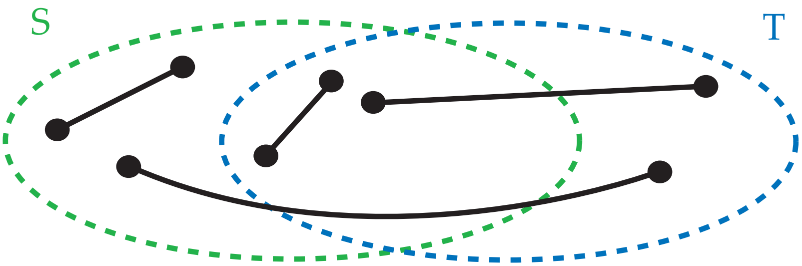
Laminar Lemma Proof

Lemma

$\sum_{e \in E(S)} x_e$ is supermodular

$$\sum_{e \in E(S)} x_e + \sum_{e \in E(T)} x_e \leq \sum_{e \in E(S \cap T)} x_e + \sum_{e \in E(S \cup T)} x_e$$



Laminar Lemma Proof

Lemma

S, T are tight, S and T cross, then $S \cap T, S \cup T$ are tight and

$$\chi_{E(S)} + \chi_{E(T)} = \chi_{E(S \cap T)} + \chi_{E(S \cup T)}$$

$$\begin{aligned}(|S| - 1) + (|T| - 1) &= (|S \cap T| - 1) + (|S \cup T| - 1) \\ &\geq \sum_{E(S \cap T)} x_e + \sum_{E(S \cup T)} x_e \qquad \text{(feasibility)} \\ &\geq \sum_{E(S)} x_e + \sum_{E(T)} x_e \qquad \text{(supermodularity)}\end{aligned}$$

Since S and T are tight, these are all equalities

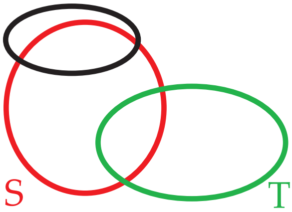
Laminar Lemma Proof: Finding $\mathcal{L}$

Lemma

$\exists\mathcal{L}$ that is laminar and $Span(\mathcal{T}) \subseteq Span(\mathcal{L})$, where $\mathcal{T}$ contains all tight sets

Let $\mathcal{L}$ be a maximal laminar collection of $\mathcal{T}$

Recall that $\chi_{E(S)} + \chi_{E(T)} = \chi_{E(S\cap T)} + \chi_{E(S\cup T)}$



$Span(\mathcal{T})$	$Span(\mathcal{L})$
S (least int. in $\mathcal{L}$)	T (int. S)
$S \cup T$ and $S \cap T$	~~$S \cup T$ and $S \cap T$~~
$S \cap T$	$S \cup T$
$S \cup T$	$S \cap T$

Laminar Lemma Proof: Finding $\mathcal{L}$

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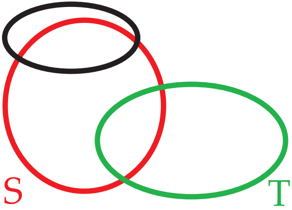

Span($\mathcal{T}$)	*Span*($\mathcal{L}$)
S (least int. in $\mathcal{L}$)	T (int. S)
$S \cup T$ and $S \cap T$	~~$S \cup T$ and $S \cap T$~~
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$S \cup T$	$S \cap T$

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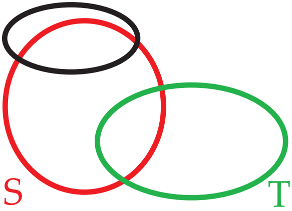

$Span(\mathcal{T})$	$Span(\mathcal{L})$
S (least int. in $\mathcal{L}$)	T (int. S)
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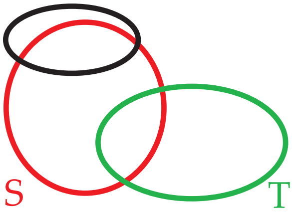

$Span(\mathcal{T})$	$Span(\mathcal{L})$
S (least int. in $\mathcal{L}$)	T (int. S)
$S \cup T$ and $S \cap T$	~~$S \cup T$ and $S \cap T$~~
$S \cap T$	$S \cup T$
$S \cup T$	$S \cap T$

Laminar Lemma Proof: Finding $\mathcal{L}$

Lemma

$\exists\mathcal{L}$ that is laminar and Span($\mathcal{T}$) $\subseteq$ Span($\mathcal{L}$), where $\mathcal{T}$ contains all tight sets

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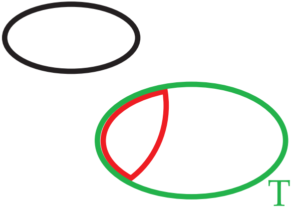

Span($\mathcal{T}$)	Span($\mathcal{L}$)
S (least int. in $\mathcal{L}$)	T (int. S)
$S \cup T$ and $S \cap T$	~~$S \cup T$ and $S \cap T$~~
$S \cap T$	$S \cup T$
$S \cup T$	$S \cap T$

Laminar Lemma Proof: Finding $\mathcal{L}$

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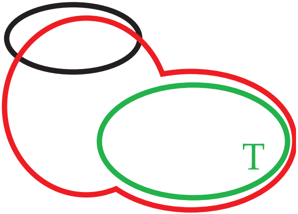

Span($\mathcal{T}$)	*Span*($\mathcal{L}$)
S (least int. in $\mathcal{L}$)	T (int. S)
$S \cup T$ and $S \cap T$	~~$S \cup T$ and $S \cap T$~~
$S \cap T$	$S \cup T$
$S \cup T$	$S \cap T$

Laminar Lemma Proof: Finding Z

Lemma

For any basic LP solution $\bar{x}$ there is a $Z \subseteq W$ and a collection $\mathcal{L}$ of $S \subseteq V$ where:

1. $\forall S \in \mathcal{L}$, *S is tight;* $\forall v \in Z$, *v is tight*
2. *The vectors* $\chi_{E(S)}$ *and* $\chi_{\delta(v)}$ *are independent*
3. $|\mathcal{L}| + |Z| = |Support(\bar{x})|$
4. $\mathcal{L}$ *is laminar*

- $(\mathcal{T}, Y)$ spans $R^{|Support(\bar{x})|}$
- $(\mathcal{L}, Y)$ spans $R^{|Support(\bar{x})|}$
- To obtain $(\mathcal{L}, Z)$ remove $v \in Y$ that are dependent

Recap

- LP formulation
- Main Theorem
- Algorithm
- Cost no more than *OPT*
- Degree bounds violated by at most 2
- Main Theorem Proof
- Laminar Lemma Proof

Outline

Improved Main Theorem

Theorem

If $\bar{x}$ is a basic solution to LP where $W \neq \emptyset$ then there is a v, s.t.

$$|\delta(v) \cap Support(\bar{x})| \leq b_v + 1$$

Algorithm

Phase 1:
While $W \neq \emptyset$

- $\bar{x} \leftarrow$ LP solution on $< G, W >$
- For all $x_e = 0$, remove e from E
- Remove v from W if there are at most $b_v + 1$ edges of $\delta(v)$ in $Support(\bar{x})$

Phase 2:
Run algorithm on $< G', \emptyset >$

Analysis

Theorem (Singh and Lau)

The improved algorithm for MBDST produces a spanning tree in which the degree of v is at most $b_v + 1$ for $v \in W$ and has cost no greater than OPT

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Thank You!

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