CS 171
Lecture Outline
February 14, 2012

Example. A transmitter sends binary bits, 80% 0's and 20% 1's. When a 0 is sent, the receiver will detect it correctly 80% of the time. When a 1 is sent, the receiver will detect it correctly 90% of the time.
(a) What is the probability that a 1 is sent and a 1 is received?
(b) If a 1 is received, what is the probability that a 1 was sent?

Solution. We will consider the following events.

S_0: event that the transmitter sent a 0.
S_1: event that the transmitter sent a 1.
R_1: event that 1 was received.

(a) We are interested in finding $\Pr[S_1 \cap R_1]$.

$$\begin{aligned} \Pr[S_1 \cap R_1] &= \Pr[S_1] \times \Pr[R_1|S_1] \\ &= 0.2 \times 0.9 \\ &= 0.18 \end{aligned}$$

(b) We are interested in finding $\Pr[S_1|R_1]$.

$$\begin{aligned} \Pr[S_1|R_1] &= \frac{\Pr[S_1 \cap R_1]}{\Pr[R_1]} \\ &= \frac{0.18}{\Pr[R_1 \cap S_1] + \Pr[R_1 \cap S_0]} \\ &= \frac{0.18}{0.18 + \Pr[S_0] \times \Pr[R_1|S_0]} \\ &= \frac{0.18}{0.18 + 0.8 \times 0.2} \\ &= 0.5294 \end{aligned}$$

Example. An urn contains 5 white and 10 black balls. A fair die is rolled and that number of balls are chosen from the urn.
(a) What is the probability that all of the balls selected are white?
(b) What is the conditional probability that the die landed on 3 if all the balls selected are white?

Solution. We will consider the following events.

W: event that all of the balls chosen are white.
D_i: event that the die landed on i, $1 \leq i \leq 6$.

(a) We want to find $\Pr[W]$. We can do this as follows.

$$\begin{aligned}
\Pr[W] &= \sum_{i=1}^{6} \Pr[W \cap D_i] \\
&= \sum_{i=1}^{6} \Pr[D_i] \Pr[W|D_i] \\
&= \sum_{i=1}^{6} \frac{1}{6} \frac{\binom{5}{i}}{\binom{15}{i}} \\
&= \frac{1}{6}\left(\frac{5}{10} + \frac{10}{105} + \frac{10}{455} + \frac{5}{1365} + \frac{1}{3003}\right) \\
&= 0.1035
\end{aligned}$$

(b) We want to find $\Pr[D_3|W]$. This can be done as follows.

$$\begin{aligned}
\Pr[D_3|W] &= \frac{D_3 \cap W}{\Pr[W]} \\
&= \frac{\Pr[D_3] \times \Pr[W|D_3]}{\Pr[W]} \\
&= \frac{1/6 \times \binom{5}{3}\binom{15}{3}}{0.1035} \\
&= \frac{1/6 \times 10/455}{0.1035} \\
&= \frac{0.00366}{0.1035} \\
&= 0.03539
\end{aligned}$$

Independent Events. Two events A and B are *independent* if $\Pr[A|B] = \Pr[A]$. This definition also implies that if A and B are independent events then $\Pr[A \cap B] = \Pr[A] \times \Pr[B]$.

Events $A_1, A_2, \ldots, A_n$ are *mutually independent* if $\forall i, 1 \leq i \leq n$ A_i does not "depend" on any combination of the other events. More formally, for every subset $I \subseteq \{1, 2, \ldots, n\} \setminus \{i\}$,

$$\Pr[A_i | \cap_{j \in I} A_j] = \Pr[A_i]$$

In other words, to show that $A_1, A_2, \ldots, A_n$ are mutually independent we must show that

all of the following hold.

$$\begin{aligned}
\Pr[A_i \cap A_j] &= \Pr[A_i] \cdot \Pr[A_j] \quad \forall \text{ distinct } i, j \\
\Pr[A_i \cap A_j \cap A_k] &= \Pr[A_i] \cdot \Pr[A_j] \cdot \Pr[A_k] \quad \forall \text{ distinct } i, j, k \\
\Pr[A_i \cap A_j \cap A_k \cap A_l] &= \Pr[A_i] \cdot \Pr[A_j] \cdot \Pr[A_k] \cdot \Pr[A_l] \quad \forall \text{ distinct } i, j, k, l \\
&\dots \\
\Pr[A_1 \cap A_2 \cap \dots \cap A_n] &= \Pr[A_1] \Pr[A_2] \cdots \Pr[A_n]
\end{aligned}$$

The above definition implies that if $A_1, A_2, \dots, A_n$ are mutually independent events then

$$\Pr[A_1 \cap A_2 \cap \dots \cap A_n] = \Pr[A_1] \times \Pr[A_2] \times \cdots \times \Pr[A_n]$$

However, note that $\Pr[A_1 \cap A_2 \cap \dots \cap A_n] = \Pr[A_1] \times \Pr[A_2] \times \cdots \times \Pr[A_n]$ is not a sufficient condition for $A_1, A_2, \dots, A_n$ to be mutually independent.

Do not confuse the concept of disjoint events and independent events. If two events A and B are disjoint and have a non-zero probability of happening then given that one event happens reduces the chances of the other event happening to zero, i.e., $\Pr[A|B] = 0 \neq \Pr[A]$. Thus by definition of independence, events A and B are not independent.

Example. Two cards are sequentially drawn (without replacement) from a well-shuffled deck of 52 cards. Let A be the event that the two cards drawn have the same value (e.g. both 4s) and let B be the event that the first card drawn is an ace. Are these events independent?

Solution. To decide whether the two events are independent we need to check whether $\Pr[AB] = \Pr[A] \Pr[B]$.

$$\begin{aligned}
\Pr[A] &= \frac{3}{51} = \frac{1}{17} \\
\Pr[B] &= \frac{4}{52} = \frac{1}{13} \\
\Pr[A \cap B] &= \frac{1}{13} \times \frac{3}{51} \\
&= \frac{1}{221} \\
&= \frac{1}{17} \times \frac{1}{13} \\
&= \Pr[A] \Pr[B]
\end{aligned}$$

Example. Suppose that a fair coin is tossed twice. Let A be the event that a head is obtained on the first toss, B be the event that a head is obtained on the second toss, and C be the event that either two heads or two tails are obtained. (a) Are events A, B, C pairwise independent? (b) Are they mutually independent?

Solution. Note that $\Omega = \{HH, HT, TH, TT\}$. $A = \{HH, HT\}$, $B = \{HH, TH\}$, $C = \{HH, TT\}$, $A \cap B = \{HH\}$, $A \cap C = \{HH\}$, $B \cap C = \{HH\}$, $A \cap B \cap C = \{HH\}$. The probabilities of the relevant events are as follows.

$$\begin{aligned}
\Pr[A] &= 1/2 \\
\Pr[B] &= 1/2 \\
\Pr[C] &= 1/2 \\
\Pr[A \cap B] &= 1/4 = \Pr[A]\Pr[B] \\
\Pr[A \cap C] &= 1/4 = \Pr[A]\Pr[C] \\
\Pr[B \cap C] &= 1/4 = \Pr[B]\Pr[C] \\
\Pr[A \cap B \cap C] &= 1/4 \neq \Pr[A]\Pr[B]\Pr[C]
\end{aligned}$$

Thus we see that A, B, C are pairwise independent but not mutually independent.