CS 171

Lecture Outline

January 31, 2012

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r-Combinations with Repetition Allowed.

We have seen that there are $\binom{n}{r}$ ways of choosing r *distinct* elements from a set of n distinct elements. What if we allow elements to be repeated? In other words, we want to find the number of ways there are to choose a multiset of r elements from a multiset of n distinct elements with infinite copies of each of the n elements available?

The following method was suggested in class.

A multiset of r elements can be constructed in r steps as follows. In Step i, choose one of the n elements. Since each step can be done in n ways, there are n^r multisets of r elements. Is this correct? No, this is not correct. For example, let $S = \{a, b\}$. Suppose we want to find the number of 2-combinations of S with repetition allowed. Note that the above procedure would consider the sets $\{a, b\}$ and $\{b, a\}$ as different whereas they are the same multiset and should not be counted twice. Using the above solution we get the answer as 4, but the correct answer is 3. In other words, the above procedure gives incorrect answer as it pays attention to the order of the r elements. We give the correct solution below.

Think of the n elements of the set as categories. Then each multiset of size r can be represented as a string of $n-1$ vertical bars (to separate the n categories) and r crosses (to represent the r elements to be chosen). The number of crosses in each category represents the number of times the element represented by that category is repeated. The number of strings of $n-1$ vertical bars and r crosses is the number of ways to choose r positions from the available $r + n - 1$ positions. The r positions chosen will contain the crosses and the remaining positions will have the vertical bars. Thus the total number of possible ways to choose multisets of size r from n elements is $\binom{r+n-1}{r}$.

Example. Consider 3 books: a computer science book, a math book, and a history book. Suppose the library has at least 6 copies of each of these books. How many ways are there to select 6 books?

Solution. The no of ways is $\binom{6+3-1}{6} = \frac{8!}{6!2!} = 28$.

Example. How many solutions are there to the equation $x_1+x_2+x_3+x_4 = 10$ if x_1, x_2, x_3, and x_4 are non-negative integers? What is each $x_i \geq 1$?

Solution. Think of x_1, x_2, x_3, and x_4 as categories in which we must place 10 $\times$'s. The number of $\times$'s in each category represents the value of the corresponding variable in the equation. The number of solutions is the number of 10 multisets of a 4-element set. This is given by

$$\binom{4+10-1}{10} = \binom{13}{10} = 286$$

If each $x_i \geq 1$, we put one $\times$ in each category to start with. Then we distribute the remaining 6 $\times$'s among the categories. Such a distribution can be represented by a string of 3 vertical bars and 6 crosses. The number of such distributions are

$$\binom{6+3}{6} = \binom{9}{6} = 84$$

Example. What is the number of non-decreasing sequences of length 10 whose terms are taken from 1 through 25?

Solution. Think of the 25 digits as 25 categories in which 10 $\times$'s are to be placed. Each arrangement of $\times$'s and $|$'s correspond to a non-decreasing sequence of length 10. The total number of ways that this can be done is given by

$$\binom{25+10-1}{10} = \binom{34}{10}$$

The Algebra of Combinations.

Example. Show that $\binom{n}{r} = \binom{n}{n-r}$.

Solution. We can ofcourse prove it algebrically. However, here is a combinatorial argument which provides more intuition. Observe that for every set of r elements that is chosen there is exactly one set of $n-r$ elements that is not chosen. Thus if a set A has k subsets of size r: $B_1, B_2, \ldots, B_k$ then each B_i can be paired up with exactly one set of size $n-r$, namely its complement $A \setminus B_i$. Hence the number of subsets of size r is same as the number of subsets of size $n-r$.

Pascal's Formula. If n and k are positive integers with $n \geq k$ then

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. Let $S = \{x_1, x_2, \ldots, x_n\}$ be the set of n elements. The k-element subsets of S can be partitioned into those that contain x_n and those that don't. In the former type of subset the other $k-1$ elements come from $S \setminus \{x_n\}$. There are $\binom{n-1}{k-1}$ ways of choosing these subsets. In the latter case all of the k elements must be chosen from $S \setminus \{x_n\}$. There are $\binom{n-1}{k}$ ways of doing this. Thus, by the addition rule the number of k-element subsets of S is given by

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The Binomial Theorem

A binomial is a sum of two terms, such as $a+b$. The *binomial theorem* gives an expression for $(a+b)^n$ where a and b are real numbers and n is a positive integer.

Theorem. For any real numbers a and b and non-negative integer n

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. Observe that each term in the expansion of $(a+b)^n$ is of the form $a^{n-k}b^k, k = 0, 1, 2, \ldots, n$. How many terms are there of the form $a^{n-k}b^k$. This is the same number of times as there are orderings of $n-k$ a's and k b's. This is equal to $\binom{n}{k}$. Thus the coefficient of like terms of the form $a^{n-k}b^k$ is $\binom{n}{k}$. This proves the theorem.

Example. Prove that $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Solution. Last week we proved this claim using a counting argument in which we showed that L.H.S. and R.H.S. count the number of subsets of a set of n elements. Now we will prove this using the binomial theorem as follows.

$$\begin{aligned}
2^n &= (1+1)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \\
&= \text{R.H.S.}
\end{aligned}$$

Example. Let n be a positive integer. Then, for all x prove that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Solution. Using the binomial theorem we get

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Example. Prove that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0$$

Solution. One way to solve this problem is by substituting $x = -1$ in the previous example. When $x = -1$ the above equation becomes

$$0^n = 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

A combinatorial argument however will give us more insight into what the expression means. We now present a combinatorial proof of the claim. We want to prove that

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$$

Consider a set $X = \{x_1, x_2, x_3, \ldots, x_n\}$. We want to show that the total number of subsets of X that have even size equals the total number of subsets of X that have odd size. We will now show that both these quantities equal 2^{n-1} from which the claim follows.
An even-sized subset of X can be constructed as follows.

Step 1. Decide whether x_1 belongs to the subset of not.
Step 2. Decide whether x_2 belongs to the subset of not.
$\vdots$
Step n. Decide whether x_n belongs to the subset of not.

Note that there are 2 choices for each of the first $n - 1$ steps but exactly one choice for performing step n. This is because if we have choose an even number of elements from $X \setminus \{x_n\}$ then we must leave out x_n otherwise we must include x_n in the subset. Hence using the multiplication rule the total number of even-sized subsets of X equals 2^{n-1}. Since we know that the total number of subsets of X is 2^n, the total number of odd-sized subsets of X is $2^n - 2^{n-1} = 2^{n-1}$.