CS 171
Lecture Outline
January 19, 2012

In the last class we did the multiplication rule and worked through some simple examples. We will work through some more examples before moving on.

Example. How many odd numbers between 1000 and 9999 have distinct digits?

Solution.
Attempt 1: An odd number from 1000 through 9999 can be constructed in four steps as follows.

Step 1. Choose the first digit.

Step 2. Choose the second digit.

Step 3. Choose the third digit.

Step 4. Choose the fourth digit.

Observe that the number of ways of performing Step 4 depends upon the choices made in the earlier steps. For example, if the choices made in the first three steps are 1, 3, and 5, then Step 4 can be performed in two ways. However, if the choices made in the first three steps are 2, 4, and 6 then Step 4 can be performed in five ways. Hence, we cannot apply multiplication rule to solve the problem in the above manner.

Attempt 2: An odd number from 1000 through 9999 can be constructed in four steps as follows.

Step 1. Choose the fourth digit.

Step 2. Choose the third digit.

Step 3. Choose the second digit.

Step 4. Choose the first digit.

Note that the number of ways of performing Step 4 depends upon whether a zero was chosen in the earlier steps. If a zero was chosen in either Step 2 or Step 3 then the number of ways of performing Step 4 is 7, otherwise it is 6. Hence, multiplication rule cannot be applied to solve the problem in the above manner.

Attempt 3. An odd number from 1000 through 9999 can be constructed in four steps as follows.

Step 1. Choose the fourth digit.

Step 2. Choose the first digit.

Step 3. Choose the second digit.

Step 4. Choose the third digit.

There are 5 ways to perform Step 1, 8 ways to perform Step 2, 8 ways to perform Step 3, and 7 ways to perform Step 4. Note that the number of ways of doing each step is independent of the choices made in the earlier steps. By the multiplication rule, the number of odd numbers from 1000 through 9999 equals $5 \times 8 \times 8 \times 7 = 2240$.

Permutations.

A permutation of a set of distinct objects is an ordering of the objects in a row. For example, the set of elements x, y, and z has six permutations: $xyz, xzy, yxz, yzx, zxy, zyx$.

In general, how many permutations are possible if we have a set of n distinct objects?

A permutation can be obtained in a sequence of n steps such that in step $i, 1 \leq i \leq n$, we choose the ith element in the ordering. Note that step i can be performed in i ways regardless of the choices made in the first $i-1$ steps. By multiplication rule, the number of permutations is

$$n \times n-1 \times n-2 \times \cdots \times 2 \times 1 = n!.$$

Example 3. Consider the set of letters $\{a, b, c, d, e, f, g, h\}$. (a) How many possible permutations are there of these letters? (b) How many permutations of these letters contain the substring abc?

Solution.
(a) There are 8 distinct elements and hence 8! permutations.
(b) We consider the string abc as one unit and that along with the remaining elements amounts to 6 distinct elements. Hence there are 6! possible permutations.

The following question was raised in class. Can we solve part (b) from first principles? We can do it as follows.
A permutation of letters consisting of substring abc can be constructed in eight steps as follows. In Steps 1, 2, and 3, choose the positions for a, b, and c, respectively. In Step i, $4 \leq i \leq 8$, choose position for the ith element in the set. Step 1 can be performed in 6 ways as a can be placed only in the first six positions. Choosing a position for a also decides positions for b and c. Hence, Steps 2 and 3 can be performed in exactly 1 way. Step i, $4 \leq i \leq 8$ can be performed in $8-(i-1)$ ways regardless of the choices made in the earlier steps. By the multiplication rule, the number of required permutations is given by

$$6 \times 1 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$$

A Brief Detour: Set Operations.

We will make a small detour to understand operations on sets. Below are some definitions.

- Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both. As an example, if $A = \{\text{Ron, Bob, Kelly}\}$ and $B = \{\text{Tim, Ryan, Bob}\}$ then $A \cup B = \{\text{Ron, Bob, Kelly, Tim, Ryan}\}$.

- Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set that contains those elements that are in both A and B. For example, if $A = \{\text{Ron, Bob, Kelly}\}$ and $B = \{\text{Tim, Ryan, Bob}\}$ then $A \cap B = \{\text{Bob}\}$.

- Two sets are called *disjoint* if their intersection is an empty set.

- A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a *partition* of a set A if, and only if, (i) $A = \bigcup_{i=1}^{n} A_i$ and (ii) $A_1, A_2, \ldots, A_n$ are mutually (pairwise) disjoint.

- Let A and B be two sets. The *difference* of A and B, denoted by $A \setminus B$ (or $A - B$) is the set containing those elements that are in A but not in B. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 6, 8\}$ then $A \setminus B = \{1\}$.

- Let A and B be sets. The *cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs formed by taking an element from A together with an element from B in all possible ways. That is, $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Permutations of Selected Elements.

We looked at permutations of n elements out of the available n elements. Now we will consider permutations of r elements out of the available n elements. Such an arrangement is called an r-permutation. For example, ab, ba, ac, ca, bc, cb are all 2-permutations of the set $\{a, b, c\}$.

Let $P(n, r)$ denote the number of r-permutations of a set of n elements. What is the value of $P(n, r)$?

Forming an r-permutation of a set of n elements can be thought of as an r-step process such that in step $i, 1 \leq i \leq r$, we choose the ith element of the ordering. There are $n - (i - 1) = n - i + 1$ ways of performing step i. By the multiplication rule, the number of r-permutations equals

$$\begin{aligned} P(n, r) &= n \times n - 1 \times n - 2 \times \cdots \times n - (r - 1) \\ &= n \times n - 1 \times n - 2 \times \cdots \times n - r + 1 \\ &= \frac{n \times (n - 1) \times \cdots \times (n - r + 1) \times (n - r) \times \cdots \times 1}{n - r \times (n - r - 1) \times (n - r - 2) \times \cdots \times 1} \\ &= \frac{n!}{(n - r)!} \end{aligned}$$

Example. How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different contestants?

Solution. Selecting the winners can be done in 3 steps with each step $i, 1 \leq i \leq 3$ choosing the winner in the ith place. Step i can be performed in $100 - (i - 1)$ ways. By multiplication rule, the total number of possible ways in which the prizes can be given is $100 \times 99 \times 98 = 970200$. Note that this is same as $P(100, 3)$.

Example. In how many ways can we order 26 letters of the alphabet so that no two of the vowels a, e, i, o, u occur consecutively?

Solution. The task of ordering the letters so that no two vowels appear consecutively can be performed in two steps.

Step 1. Order the 21 consonants.

Step 2. Choose locations for the 5 vowels. The vowels can be placed before the consonants, between the consonants and after the consonants.

Step 1 can be performed in 21! ways. To count the number of ways of performing Step 2, observe that there is only one location for placing a vowel before and after the consonants, and 20 locations for placing the vowels between the consonants. This gives a total of 22 valid locations for placing 5 vowels. Thus the number of ways of placing the 5 vowels in 5 of the 22 locations is $P(22, 5)$. This is because there are 22 locations for a, 21 for e, 20 for i, 19 for o, and 18 for u. By multiplication rule, the total number of orderings in which no two vowels occur consecutively equals

$$21! \times P(22, 5) = \frac{21! \times 22!}{17!}$$