Math 314 EXAM 2 SAMPLE QUESTIONS

— ANSWERS —

1) $3\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$ is equivalent to

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 0$$

To convert, set $v = \frac{dy}{dt}$, so that we have (by differentiating) $\frac{dv}{dt} = \frac{d^2y}{dt^2}$.

An equivalent first order system is

$$\boxed{\begin{aligned} \frac{dy}{dt} &= v, \\ \frac{dv}{dt} &= -2v - 3y. \end{aligned}}$$

or, with $\underset{\sim}{Y}(t) = \begin{pmatrix} y(t) \\ v(t) \end{pmatrix}$, the equivalent system can be written

$$\boxed{\frac{d\underset{\sim}{Y}}{dt} = \underset{\sim}{A}\underset{\sim}{Y} \quad \text{with } \underset{\sim}{A} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}}.$$

2) e-values of $\underset{\sim}{A} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix}$:

$$\det(\underset{\sim}{A} - \lambda \underset{\sim}{I}) = \det\begin{pmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix}$$

$$= (-2-\lambda)(1-\lambda) - 4 = (-2 + 2\lambda - \lambda + \lambda^2) - 4$$

$$= \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

e-values are the solutions (roots) of

$$(\lambda+3)(\lambda-2) = 0,$$

namely, $\lambda_1 = -3$ and $\lambda_2 = 2$.

e-vectors :

for $\lambda_1 = -3$, $\underset{\sim}{A}\vec{v}_1 = \lambda_1 \vec{v}_1$ or $(\underset{\sim}{A} - \lambda_1 I)\vec{v}_1 = 0$

$$\iff \left(\begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} - (-3)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} -2+3 & -2 \\ -2 & 1+3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \left\{ \begin{array}{l} x_1 - 2y_1 = 0 \\ -2x_1 + 4y_1 = 0 \end{array} \right\} \iff x_1 = 2y_1 \quad \left(\text{or } y_1 = \tfrac{1}{2}x_1\right)$$

e.g., $\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 2$, $\underset{\sim}{A}\vec{v}_2 = \lambda_2 \vec{v}_2$ or $(\underset{\sim}{A} - \lambda_2 \underset{\sim}{I})\vec{v}_2 = \vec{0}$

$$\iff \left(\begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} -2-2 & -2 \\ -2 & 1-2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \left\{ \begin{array}{l} -4x_2 - 2y_2 = 0 \\ -2x_2 - y_2 = 0 \end{array} \right\} \iff -2x_2 = y_2$$

e.g., $\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$$\therefore \boxed{\underset{\sim}{Y}(t) = k_1 e^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ is the general solution.}}$$

3) A good guess is $y(t) = e^{at}$ for some suitable constant $\underline{a}$. To find all suitable values for the variable (or parameter) $\underline{a}$ we substitute e^{at} for $y(t)$ in the differential equation to get

$$2a^2 e^{at} + 7ae^{at} + 3e^{at} = 0 \qquad (1)$$

(since $\frac{dy}{dt} = \frac{d}{dt}e^{at} = ae^{at}$)

Factor out e^{at}, which is never zero:

$$2a^2 + 7a + 3 = 0 \qquad (2)$$

Solve equation (2) to get

$$a = \frac{-7 \pm \sqrt{49 - 24}}{4} = \frac{-7 \pm \sqrt{25}}{4} = \frac{-7 \pm 5}{4}$$

$$\Rightarrow a \in \{a_1, a_2\} \text{ where } \begin{cases} a_1 = \frac{-7+5}{4} = -\frac{2}{4} = -\frac{1}{2} \\ a_2 = \frac{-7-5}{4} = \frac{-12}{4} = -3 \end{cases}$$

giving solutions

$$\boxed{y_1(t) = e^{-\frac{1}{2}t} \quad \text{and } y_2(t) = e^{-3t}}$$

for the differential equation.

4) (i) $\frac{dx}{dt} = 10x\left(1 - \frac{x}{10}\right) - 20xy$

$\frac{dy}{dt} = -5y + \frac{xy}{20}$

Equilibrium Points?

Solve

$10x - x^2 - 20xy = 0$

$-5y + \frac{xy}{20} = 0$

(ii) $\frac{dx}{dt} = 0.3x - \frac{xy}{100}$

$\frac{dy}{dt} = 15y\left(1 - \frac{y}{15}\right) + 25xy$

Solve

$0.3x - \frac{xy}{100} = 0$

$15y\left(1 - \frac{y}{15}\right) + 25xy = 0$

4) continued –

(i) Equivalently, we have

$$(10 - x - 20y)x = 0$$
$$\left(-5 + \frac{x}{20}\right)y = 0$$

That is, we have equations of the form $AB = 0$ and $CD = 0$, so solutions ought to arise from setting $A=0$ and $C=0$,
$A=0$ and $D=0$,
$B=0$ and $C=0$,
$B=0$ and $D=0$.

- If we set $10 - x - 20y = 0 = -5 + \frac{x}{20}$, then $x = 100$ and $y = \frac{10-x}{20} = \frac{-90}{20} = \frac{-9}{2}$, i.e., $\boxed{(x,y) = (100, -9/2)}$
- If we set $10 - x - 20y = 0$ and $y = 0$, then $x = 10$, and $\boxed{(x,y) = (10,0)}$.
- If we set $x = 0$ and $-5 + \frac{x}{20} = 0$ we have a contradiction.
- If we set $x=0$ and $y=0$, we have equilibrium solution $\boxed{(x,y) = (0,0)}$.

4) continued -

(ii) $\begin{cases} 0.3\,x - \frac{xy}{100} = 0 \iff \left(0.3 - \frac{y}{100}\right)x = 0 \\ 15y\left(1 - \frac{y}{15}\right) + 25xy = 0 \iff (15 - y + 25x)\,y = 0 \end{cases}$

- $\begin{Bmatrix} 0.3 - \frac{y}{100} = 0 \\ 15 - y + 25x = 0 \end{Bmatrix} \Rightarrow y = 30,\ x = \frac{15}{25} = \frac{3}{5} \Rightarrow \boxed{(x, y) = \left(\frac{3}{5}, 30\right)}$

- $\begin{Bmatrix} 0.3 - \frac{y}{100} = 0 \\ y = 0 \end{Bmatrix}$ gives a contradiction

- $\begin{Bmatrix} x = 0 \\ 15 - y + 25x = 0 \end{Bmatrix} \Rightarrow y = 15 \Rightarrow \boxed{(x, y) = (0, 15)}$

- $\begin{Bmatrix} x = 0 \\ y = 0 \end{Bmatrix} \Rightarrow \boxed{(x, y) = (0, 0)}$

There is no significance for $(100, -9/2)$ in (i). For $(x, y) = (10, 0)$ there are prey but no predators. For $(x, y) = (0, 0)$ there neither prey nor predators

In (ii), for $(0, 15)$ there are predators but the prey is extinct. For $(x, y) = \left(\frac{3}{5}, 30\right)$, predators and prey coexist. For $(x, y) = (0, 0)$ there are neither predators nor prey.

5) (Sec. 2.1 #20) $\frac{d^2y}{dt^2} + \frac{k}{m} y = 0 \qquad (*)$

(a) $y(t) = \cos \beta t$. When is this a solution?

Substitute $\cos \beta t$ into $(*)$

$$0 = \frac{d^2y}{dt^2} + \frac{k}{m} y = \frac{d^2}{dt^2}(\cos \beta t) + \frac{k}{m} \cos \beta t$$

$$= -\beta^2 \cos \beta t + \frac{k}{m} \cos \beta t$$

$$= \left(\frac{k}{m} - \beta^2\right) \cos \beta t .$$

In order for $y(t) = \cos \beta t$ to be a solution of $(*)$ we require $\frac{k}{m} - \beta^2 = 0$, so

$$\boxed{\beta = \pm\sqrt{\frac{k}{m}}}$$

(b) What initial condition $(t=0)$ in the yv-plane corresponds to ~~this~~ solution?

First, notice that $v = \frac{dy}{dt} = \frac{d}{dt} \cos \beta t$, so we have $v = -\beta \sin \beta t$.

The system

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -\frac{k}{m} y$$

has solution $\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} \cos \beta t \\ -\beta \sin \beta t \end{pmatrix}$

corresponding to $y(t)$, and at $t=0$

$$\begin{pmatrix} y(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} \cos 0 \\ -\beta \sin 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so $\boxed{y(0) = 1 \text{ and } v(0) = 0}$.

5) continued —

(c) In terms of k and m, what is the period of this solution?

The period of a function $f(t)$ is the smallest positive number p such that $f(t+p) = f(t)$ for every t.

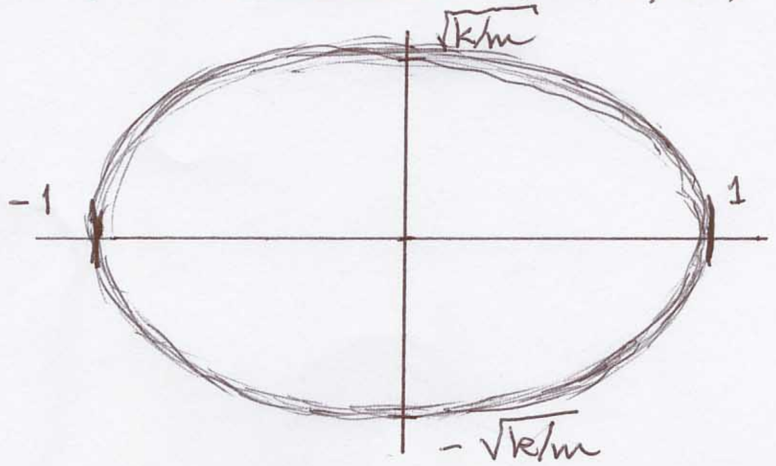
The solution $y(t) = \cos \beta t = \cos \sqrt{\frac{k}{m}}\, t$ has period $\boxed{2\pi/\sqrt{k/m} = 2\pi\sqrt{m}/\sqrt{k}}$.

(d) Sketch the solution curve (in the yv-plane) associated to the solution $y(t)$ [Hint: consider the quantity $y^2 + (v/\beta)^2$].

We observe that $y^2 + (v/\beta)^2$

$$= \cos^2 \beta t + (-\beta \sin(\beta t)/\beta)^2$$
$$= \cos^2 \beta t + \sin^2 \beta t = 1,$$

so (y, v) moves along an ellipse centered at $(0,0)$



6) §2.2 #12 – 16

#12 The system $\begin{cases} \frac{dR}{dt} = 2R(1-\frac{R}{2}) - 1.2RF \\ \frac{dF}{dt} = -F + 0.9RF, \end{cases}$

has equilibrium solutions for which

$$\begin{cases} 2R(1-\frac{R}{2}) - 1.2RF = 0 \\ -F + 0.9RF = 0. \end{cases}$$

The bottom equation holds true if $F=0$ or if $R = \frac{10}{9}$ (since $-F+0.9RF = -F(1-0.9R)$).

Let's deal separately with these cases:

(i) $F=0$. Then the top equation,

$$2R(1-\frac{R}{2}) - 1.2RF = 0$$

simplifies to

$$2R(1-\frac{R}{2}) = 0,$$

so it holds iff $R=0$ or $R=2$.

We thus find two equilibrium solutions $\boxed{(R,F)=(0,0)}$ and $\boxed{(R,F)=(2,0)}$.

(ii) $R = 10/9$. Then the top equation

$$2R(1-\frac{R}{2}) - 1.2RF = 0$$

simplifies to

$$2(1-\frac{10/9}{2}) - 1.2F = 0; \quad \text{i.e.,}$$

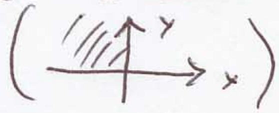
$$\frac{8}{9} - \frac{6}{5}F = 0; \quad \text{so}$$

$$F = \frac{40}{54} = \frac{20}{27}; \quad \boxed{(R,F) = \left(\frac{10}{9}, \frac{20}{27}\right)}$$

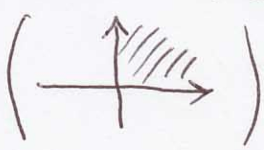
6) #14 Continued —

Graphs are <u>not</u> <u>periodic</u>.

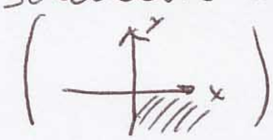
(i) Value of $x(t)$ starts negative
Value of $y(t)$ starts positive
$\therefore$ solution starts in the 2nd quadrant



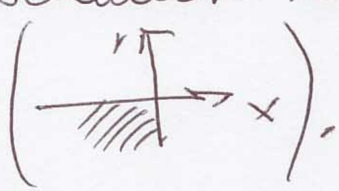
(ii) For an interval of time, both $x(t)$ and $y(t)$ are positive, so the solution curve moves into the first quadrant



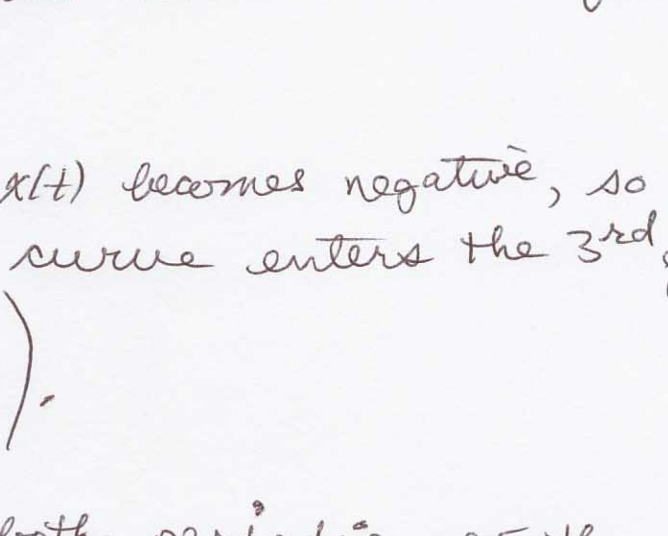
(iii) Next, $y(t)$ becomes negative, so the solution curve enters the 4th quadrant



(iv) Finally, $x(t)$ becomes negative, so the solution curve enters the 3rd quadrant

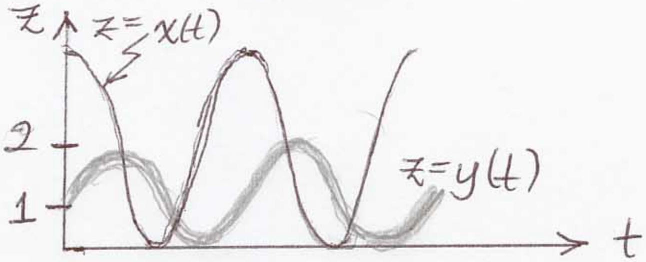


#16 $x(t)$, $y(t)$ both periodic, so the solution curve returns to its starting point.

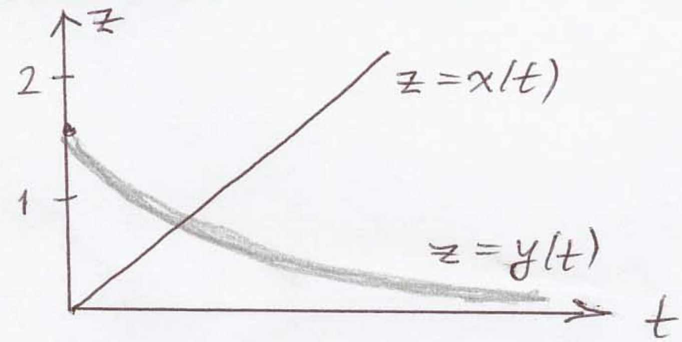


7) Sec. 2.2 # 23-26

#24 Solution curve is an ellipse centered at (2, 1), so the $x(t)$-graph is periodic, oscillates about the line $x=2$, and the $y(t)$-graph is periodic, oscillates about the line $y=1$



#26 The $x(t)$-graph starts with a small positive value and increases (monotonically) with t.
The $y(t)$-graph starts with value around 1.6 and decreases as t increases. One possible sketch -



8) §2.4 #2(a,b)

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = y$$

(a) Check that $\mathbf{Y}(t) = \begin{pmatrix} e^{2t} \\ 3e^{t} \end{pmatrix}$ is a solution.

We compute

$$\frac{dx}{dt} = \frac{d}{dt}\left(e^{2t}\right) = 2e^{2t} = 2x, \text{ so the top}$$

8) continued —
equation is satisfied. We also compute
$$\frac{dy}{dt} = \frac{d}{dt}(3e^t) = 3e^t = y,$$
so the bottom equation is also satisfied.
$\therefore$ $\underset{\sim}{Y}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{2t} \\ 3e^t \end{pmatrix}$ is a solution. // (a)

(b), (c) — See Homework #10.

Problems 9 through 14 are answered in your textbook (at the back)