CHAPTER 17

Linear Programming: Simplex Method

CONTENTS

In Chapter 2 we showed how the graphical solution procedure can be used to solve linear programming problems involving two decision variables. However, most linear programming problems are too large to be solved graphically, and an algebraic solution procedure must be employed. The most widely used algebraic procedure for solving linear programming problems is called the **simplex method.**[1] Computer programs based on this method can routinely solve linear programming problems with thousands of variables and constraints. The Management Science in Action, Fleet Assignment at Delta Air Lines, describes solving a linear program involving 60,000 variables and 40,000 constraints on a daily basis.

MANAGEMENT SCIENCE IN ACTION

FLEET ASSIGNMENT AT DELTA AIR LINES*

Delta Air Lines uses linear and integer programming in its Coldstart project to solve its fleet assignment problem. The problem is to match aircraft to flight legs and fill seats with paying passengers. Airline profitability depends on being able to assign the right size of aircraft to the right leg at the right time of day. An airline seat is a perishable commodity; once a flight takes off with an empty seat the profit potential of that seat is gone forever. Primary objectives of the fleet assignment model are to minimize operating costs and lost passenger revenue. Constraints are aircraft availability, balancing arrivals and departures at airports, and maintenance requirements.

The successful implementation of the Coldstart model for assigning fleet types to flight legs shows the size of linear programs that can be solved today. The typical size of the daily Coldstart model is about 60,000 variables and 40,000 constraints. The first step in solving the fleet assignment problem is to solve the model as a linear program. The model developers report successfully solving these problems on a daily basis and contend that use of the Coldstart model will save Delta Air Lines $300 million over the next three years.

*Based on R. Subramanian, R. P. Scheff, Jr., J. D. Quillinan, D. S. Wiper, and R. E. Marsten, “Coldstart: Fleet Assignment at Delta Air Lines,” *Interfaces* (January/February 1994): 104–120.

17.1 AN ALGEBRAIC OVERVIEW OF THE SIMPLEX METHOD

Let us introduce the problem we will use to demonstrate the simplex method. HighTech Industries imports electronic components that are used to assemble two different models of personal computers. One model is called the Deskpro, and the other model is called the Portable. HighTech’s management is currently interested in developing a weekly production schedule for both products.

The Deskpro generates a profit contribution of $50 per unit, and the Portable generates a profit contribution of $40 per unit. For next week’s production, a maximum of 150 hours of assembly time can be made available. Each unit of the Deskpro requires 3 hours of assembly time, and each unit of the Portable requires 5 hours of assembly time. In addition, HighTech currently has only 20 Portable display components in inventory; thus, no more than 20 units of the Portable may be assembled. Finally, only 300 square feet of warehouse space can be made available for new production. Assembly of each Deskpro requires 8 square feet of warehouse space; similarly, each Portable requires 5 square feet.

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[1] Several computer codes also employ what are called interior point solution procedures. They work well on many large problems, but the simplex method is still the most widely used solution procedure.

To develop a linear programming model for the HighTech problem, we will use the following decision variables:

$$\begin{aligned} x_1 &= \text{number of units of the Deskpro} \\ x_2 &= \text{number of units of the Portable} \end{aligned}$$

The complete mathematical model for this problem is presented here.

$$\begin{array}{lrcll} \text{Max} & 50x_1 + 40x_2 & & & \\ \text{s.t.} & & & & \\ & 3x_1 + 5x_2 & \leq & 150 & \text{Assembly time} \\ & 1x_2 & \leq & 20 & \text{Portable display} \\ & 8x_1 + 5x_2 & \leq & 300 & \text{Warehouse capacity} \\ & x_1, x_2 \geq 0 & & & \end{array}$$

Adding a slack variable to each of the constraints permits us to write the problem in standard form.

$$\text{Max} \quad 50x_1 + 40x_2 + 0s_1 + 0s_2 + 0s_3 \tag{17.1}$$

s.t.

$$3x_1 + 5x_2 + 1s_1 = 150 \tag{17.2}$$

$$1x_2 + 1s_2 = 20 \tag{17.3}$$

$$8x_1 + 5x_2 + 1s_3 = 300 \tag{17.4}$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0 \tag{17.5}$$

The simplex method was developed by George Dantzig while working for the U.S. Air Force. It was first published in 1949.

Algebraic Properties of the Simplex Method

Constraint equations (17.2) to (17.4) form a system of three simultaneous linear equations with five variables. Whenever a system of simultaneous linear equations has more variables than equations, we can expect an infinite number of solutions. The simplex method can be viewed as an algebraic procedure for finding the best solution to such a system of equations. In the preceding example, the best solution is the solution to equations (17.2) to (17.4) that maximizes the objective function (17.1) and satisfies the nonnegativity conditions given by (17.5).

Determining a Basic Solution

For the HighTech Industries constraint equations, which have more variables (five) than equations (three), the simplex method finds solutions for these equations by assigning zero values to two of the variables and then solving for the values of the remaining three variables. For example, if we set $x_2 = 0$ and $s_1 = 0$, the system of constraint equations becomes

$$3x_1 = 150 \tag{17.6}$$

$$1s_2 = 20 \tag{17.7}$$

$$8x_1 + 1s_3 = 300 \tag{17.8}$$

Using equation (17.6) to solve for x_1, we have

$$3x_1 = 150$$

and hence $x_1 = 150/3 = 50$. Equation (17.7) provides $s_2 = 20$. Finally, substituting $x_1 = 50$ into equation (17.8) results in

$$8(50) + 1s_3 = 300$$

Solving for s_3, we obtain $s_3 = -100$.

A basic solution is obtained by setting two of the five variables equal to zero and solving the three equations simultaneously for the values of the other three variables. Mathematically, we are guaranteed a solution only if the resulting three equations are linearly independent. Fortunately, the simplex method is designed to guarantee that a solution exists for the basic variables at each iteration.

Thus, we obtain the following solution to the three-equation, five-variable set of linear equations:

$$\begin{aligned} x_1 &= 50 \\ x_2 &= 0 \\ s_1 &= 0 \\ s_2 &= 20 \\ s_3 &= -100 \end{aligned}$$

This solution is referred to as a **basic solution** for the HighTech linear programming problem. To state a general procedure for determining a basic solution, we must consider a standard-form linear programming problem consisting of n variables and m linear equations, where n is greater than m.

Basic Solution

To determine a basic solution, set $n - m$ of the variables equal to zero, and solve the m linear constraint equations for the remaining m variables.[2]

In terms of the HighTech problem, a basic solution can be obtained by setting any two variables equal to zero and then solving the system of three linear equations for the remaining three variables. We shall refer to the $n - m$ variables set equal to zero as the **nonbasic variables** and the remaining m variables as the **basic variables.** Thus, in the preceding example, x_2 and s_1 are the nonbasic variables, and x_1, s_2, and s_3 are the basic variables.

Basic Feasible Solution

A basic solution can be either feasible or infeasible. A **basic feasible solution** is a basic solution that also satisfies the nonnegativity conditions. The basic solution found by setting x_2 and s_1 equal to zero and then solving for x_1, s_2, and s_3 is not a basic feasible solution because $s_3 = -100$. However, suppose that we had chosen to make x_1 and x_2 nonbasic variables by setting $x_1 = 0$ and $x_2 = 0$. Solving for the corresponding basic solution is easy because with $x_1 = x_2 = 0$, the three constraint equations reduce to

$$\begin{aligned} 1s_1 \qquad\qquad &= 150 \\ 1s_2 \qquad &= 20 \\ 1s_3 &= 300 \end{aligned}$$

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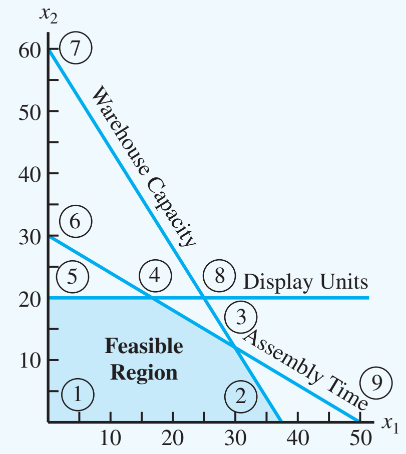
[2] In some cases, a unique solution cannot be found for a system of m equations and n variables. However, these cases will never be encountered when using the simplex method.

The complete solution with $x_1 = 0$ and $x_2 = 0$ is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ s_1 &= 150 \\ s_2 &= 20 \\ s_3 &= 300 \end{aligned}$$

This solution is a basic feasible solution because all of the variables satisfy the nonnegativity conditions.

The following graph shows all the constraint equations and basic solutions for the HighTech problem. Circled points ①–⑤ are basic feasible solutions; circled points ⑥–⑨ are basic solutions that are not feasible. The basic solution found by setting $x_2 = 0$ and $s_1 = 0$ corresponds to point ⑨; the basic feasible solution found by setting $x_1 = 0$ and $x_2 = 0$ corresponds to point ① in the feasible region.



Can you find basic and basic feasible solutions to a system of equations at this point? Try Problem 1.

The graph in Figure 17.1 shows only the basic feasible solutions for the HighTech problem; note that each of these solutions is an extreme point of the feasible region. In Chapter 2 we showed that the optimal solution to a linear programming problem can be found at an extreme point. Because every extreme point corresponds to a basic feasible solution, we can now conclude that the HighTech problem does have an optimal basic feasible solution.[3] The simplex method is an iterative procedure for moving from one basic feasible solution (extreme point) to another until the optimal solution is reached.

17.2 TABLEAU FORM

A basic feasible solution to the system of m linear constraint equations and n variables is required as a starting point for the simplex method. The purpose of tableau form is to provide an initial basic feasible solution.

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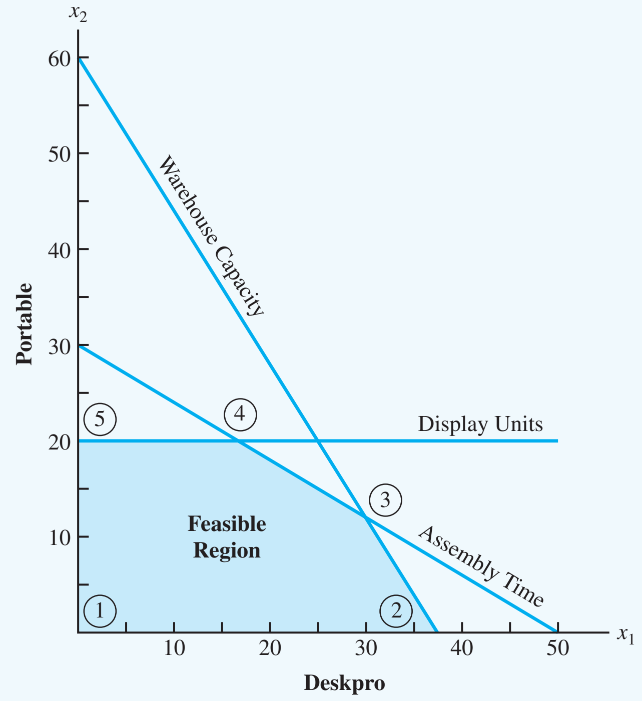
[3] We are only considering cases that have an optimal solution. That is, cases of infeasibility and unboundedness will have no optimal solution, so no optimal basic feasible solution is possible.

FIGURE 17.1 FEASIBLE REGION AND EXTREME POINTS FOR THE HIGHTECH INDUSTRIES PROBLEM



Recall that for the HighTech problem, the standard-form representation is

$$
\begin{array}{ll}
\text{Max} & 50x_1 + 40x_2 + 0s_1 + 0s_2 + 0s_3 \\
\text{s.t.} & \\
& 3x_1 + 5x_2 + 1s_1 \qquad\qquad\qquad = 150 \\
& \qquad\quad 1x_2 \qquad + 1s_2 \qquad\quad = \ \ 20 \\
& 8x_1 + 5x_2 \qquad\qquad\qquad + 1s_3 = 300 \\
& x_1, x_2, s_1, s_2, s_3 \geq 0
\end{array}
$$

When a linear programming problem with all less-than-or-equal-to constraints is written in standard form, it is easy to find a basic feasible solution. We simply set the decision variables equal to zero and solve for the values of the slack variables. Note that this procedure sets the values of the slack variables equal to the right-hand-side values of the constraint equations. For the HighTech problem, we obtain $x_1 = 0$, $x_2 = 0$, $s_1 = 150$, $s_2 = 20$, and $s_3 = 300$ as the initial basic feasible solution.

If we study the standard-form representation of the HighTech constraint equations closely, we can identify two properties that make it possible to find an initial basic feasible solution. The first property requires that the following conditions be satisfied:

a. For each constraint equation, the coefficient of one of the m basic variables in that equation must be 1, and the coefficients for all the remaining basic variables in that equation must be 0.
b. The coefficient for each basic variable must be 1 in only one constraint equation.

When these conditions are satisfied, exactly one basic variable with a coefficient of 1 is associated with each constraint equation, and for each of the m constraint equations, it is a different basic variable. Thus, if the $n - m$ nonbasic variables are set equal to zero, the values of the basic variables are the values of the right-hand sides of the constraint equations.

For linear programs with less-than-or-equal-to constraints, the slack variables provide the initial basic feasible solution identified in tableau form.

The second property that enables us to find a basic feasible solution requires the values of the right-hand sides of the constraint equations be nonnegative. This nonnegativity ensures that the basic solution obtained by setting the basic variables equal to the values of the right-hand sides will be feasible.

If a linear programming problem satisfies these two properties, it is said to be in **tableau form.** Thus, we see that the standard-form representation of the HighTech problem is already in tableau form. In fact, standard form and tableau form for linear programs that have all less-than-or-equal-to constraints and nonnegative right-hand-side values are the same. Later in this chapter we will show how to set up the tableau form for linear programming problems where the standard form and the tableau form are not the same.

In the HighTech problem, tableau form and standard form are the same, which is true for all LPs with only less-than-or-equal-to constraints and nonnegative right-hand sides.

To summarize, the following three steps are necessary to prepare a linear programming problem for solution using the simplex method:

Step 1. Formulate the problem.
Step 2. Set up the standard form by adding slack and/or subtracting surplus variables.
Step 3. Set up the tableau form.

17.3 SETTING UP THE INITIAL SIMPLEX TABLEAU

After a linear programming problem has been converted to tableau form, we have an initial basic feasible solution that can be used to begin the simplex method. To provide a convenient means for performing the calculations required by the simplex method, we will first develop what is referred to as the initial **simplex tableau.**

Part of the initial simplex tableau is a table containing all the coefficients shown in the tableau form of a linear program. If we adopt the general notation

c_j = objective function coefficient for variable j
b_i = right-hand-side value for constraint i
a_{ij} = coefficient associated with variable j in constraint i

we can show this portion of the initial simplex tableau as follows:

c_1	c_2	$\ldots c_n$	
a_{11}	a_{12}	$\ldots a_{1n}$	b_1
a_{21}	a_{22}	$\ldots a_{2n}$	b_2
.	.	. . . .	.
.	.	. . . .	.
a_{m1}	a_{m2}	$\ldots a_{mn}$	b_m

Thus, for the HighTech problem we obtain the following partial initial simplex tableau:

50	40	0	0	0	
3	5	1	0	0	**150**
0	1	0	1	0	**20**
8	5	0	0	1	**300**

Later we may want to refer to the objective function coefficients, all the right-hand-side values, or all the coefficients in the constraints as a group. For such groupings, we will find the following general notation helpful:

c row = row of objective function coefficients
b column = column of right-hand-side values of the constraint equations
A matrix = m rows and n columns of coefficients of the variables in the constraint equations

Using this notation, we can show these portions of the initial simplex tableau as follows:

c row	
A matrix	b column

To practice setting up the portion of the simplex tableau corresponding to the objective function and constraints at this point, try Problem 4.

To help us recall that each of the columns contains the coefficients for one of the variables, we write the variable associated with each column directly above the column. By adding the variables we obtain

x_1	x_2	s_1	s_2	s_3	
50	40	0	0	0	
3	5	1	0	0	**150**
0	1	0	1	0	**20**
8	5	0	0	1	**300**

This portion of the initial simplex tableau contains the tableau-form representation of the problem; thus, it is easy to identify the initial basic feasible solution. First, we note that for each basic variable, a corresponding column has a 1 in the only nonzero position. Such columns are known as **unit columns** or **unit vectors.** Second, a row of the tableau is associated with each basic variable. This row has a 1 in the unit column corresponding to the basic variable. The value of each basic variable is then given by the b_i value in the row associated with the basic variable. In the example, row 1 is associated with basic variable s_1 because this row has a 1 in the unit column corresponding to s_1. Thus, the value of s_1 is given by the right-hand-side value b_1: $s_1 = b_1 = 150$. In a similar fashion, $s_2 = b_2 = 20$, and $s_3 = b_3 = 300$.

To move from an initial basic feasible solution to a better basic feasible solution, the simplex method must generate a new basic feasible solution that yields a better value for the objective function. To do so requires changing the set of basic variables: we select one of the current nonbasic variables to be made basic and one of the current basic variables to be made nonbasic.

For computational convenience, we will add two new columns to the simplex tableau. One column is labeled "*Basis*" and the other column is labeled "c_B." In the ***Basis*** column, we list the current basic variables, and in the c_B column, we list the corresponding objective function coefficient for each of the basic variables. For the HighTech problem, this results in the following:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
s_1	0	3	5	1	0	0	**150**
s_2	0	0	1	0	1	0	**20**
s_3	0	8	5	0	0	1	**300**

Note that in the column labeled *Basis*, s_1 is listed as the first basic variable because its value is given by the right-hand-side value for the first equation. With s_2 listed second and s_3 listed third, the *Basis* column and right-hand-side values show the initial basic feasible solution has $s_1 = 150$, $s_2 = 20$, and $s_3 = 300$.

Can we improve the value of the objective function by moving to a new basic feasible solution? To find out whether it is possible, we add two rows to the bottom of the tableau. The first row, labeled z_j, represents the decrease in the value of the objective function that will result if one unit of the variable corresponding to the jth column of the A matrix is brought into the basis. The second row, labeled $c_j - z_j$, represents the net change in the value of the objective function if one unit of the variable corresponding to the jth column of the A matrix is brought into the solution. We refer to the $c_j - z_j$ row as the **net evaluation row.**

Let us first see how the entries in the z_j row are computed. Suppose that we consider increasing the value of the nonbasic variable x_1 by one unit—that is, from $x_1 = 0$ to $x_1 = 1$. In order to make this change and at the same time continue to satisfy the constraint equations, the values of some of the other variables will have to be changed. As we will show, the simplex method requires that the necessary changes be made to basic variables only. For example, in the first constraint we have

$$3x_1 + 5x_2 + 1s_1 = 150$$

The current basic variable in this constraint equation is s_1. Assuming that x_2 remains a nonbasic variable with a value of 0, if x_1 is increased in value by 1, then s_1 must be decreased by 3 for the constraint to be satisfied. Similarly, if we were to increase the value of x_1 by 1 (and keep $x_2 = 0$), we can see from the second and third equations that although s_2 would not decrease, s_3 would decrease by 8.

From analyzing all the constraint equations, we see that the coefficients in the x_1 column indicate the amount of decrease in the current basic variables when the nonbasic variable x_1 is increased from 0 to 1. In general, all the column coefficients can be interpreted this way. For instance, if we make x_2 a basic variable at a value of 1, s_1 will decrease by 5, s_2 will decrease by 1, and s_3 will decrease by 5.

Recall that the values in the c_B column of the simplex tableau are the objective function coefficients for the current basic variables. Hence, to compute the values in the z_j row (the decrease in value of the objective function when x_j is increased by one), we form the sum of the products obtained by multiplying the elements in the c_B column by the corresponding elements in the jth column of the A matrix. Doing these calculations we obtain

$$\begin{aligned}
z_1 &= 0(3) + 0(0) + 0(8) = 0\\
z_2 &= 0(5) + 0(1) + 0(5) = 0\\
z_3 &= 0(1) + 0(0) + 0(0) = 0\\
z_4 &= 0(0) + 0(1) + 0(0) = 0\\
z_5 &= 0(0) + 0(0) + 0(1) = 0
\end{aligned}$$

Because the objective function coefficient of x_1 is $c_1 = 50$, the value of $c_1 - z_1$ is $50 - 0 = 50$. Then the net result of bringing one unit of x_1 into the current basis will be an increase in profit of \$50. Hence, in the net evaluation row corresponding to x_1, we enter 50. In the same manner, we can calculate the $c_j - z_j$ values for the remaining variables. The result is the following initial simplex tableau:

The simplex tableau is nothing more than a table that helps keep track of the simplex method calculations. Reconstructing the original problem can be accomplished from the initial simplex tableau.

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
s_1	0	3	5	1	0	0	**150**
s_2	0	0	1	0	1	0	**20**
s_3	0	8	5	0	0	1	**300**
z_j		0	0	0	0	0	**0**
$c_j - z_j$		50	40	0	0	0	↑

Value of the Objective Function

In this tableau we also see a boldfaced 0 in the z_j row in the last column. This zero is the value of the objective function associated with the current basic feasible solution. It was computed by multiplying the objective function coefficients in the c_B column by the corresponding values of the basic variables shown in the last column of the tableau—that is, $0(150) + 0(20) + 0(300) = 0$.

Try Problem 5(a) for practice in setting up the complete initial simplex tableau for a problem with less-than-or-equal-to constraints.

The initial simplex tableau is now complete. It shows that the initial basic feasible solution ($x_1 = 0, x_2 = 0, s_1 = 150, s_2 = 20$, and $s_3 = 300$) has an objective function value, or profit, of \$0. In addition, the $c_j - z_j$ or net evaluation row has values that will guide us in improving the solution by moving to a better basic feasible solution.

17.4 IMPROVING THE SOLUTION

From the net evaluation row, we see that each unit of the Deskpro (x_1) increases the value of the objective function by 50 and each unit of the Portable (x_2) increases the value of the objective function by 40. Because x_1 causes the largest per-unit increase, we choose it as the variable to bring into the basis. We must next determine which of the current basic variables to make nonbasic.

In discussing how to compute the z_j values, we noted that each of the coefficients in the x_1 column indicates the amount of decrease in the corresponding basic variable that would result from increasing x_1 by one unit. Considering the first row, we see that every unit of the Deskpro produced will use 3 hours of assembly time, reducing s_1 by 3. In the current solution, $s_1 = 150$ and $x_1 = 0$. Thus—considering this row only—the maximum possible value of x_1 can be calculated by solving

$$3x_1 = 150$$

which provides

$$x_1 = 50$$

If x_1 is 50 (and x_2 remains a nonbasic variable with a value of 0), s_1 will have to be reduced to zero in order to satisfy the first constraint:

$$3x_1 + 5x_2 + 1s_1 = 150$$

Considering the second row, $0x_1 + 1x_2 + 1s_2 = 20$, we see that the coefficient of x_1 is 0. Thus, increasing x_1 will not have any effect on s_2; that is, increasing x_1 cannot drive the basic variable in the second row (s_2) to zero. Indeed, increases in x_1 will leave s_2 unchanged.

Finally, with 8 as the coefficient of x_1 in the third row, every unit that we increase x_1 will cause a decrease of eight units in s_3. Because the value of s_3 is currently 300, we can solve

$$8x_1 = 300$$

to find the maximum possible increase in x_1 before s_3 will become nonbasic at a value of zero; thus, we see that x_1 cannot be any larger than $^{300}\!/_8 = 37.5$.

Considering the three rows (constraints) simultaneously, we see that row 3 is the most restrictive. That is, producing 37.5 units of the Deskpro will force the corresponding slack variable to become nonbasic at a value of $s_3 = 0$.

In making the decision to produce as many Deskpro units as possible, we must change the set of variables in the basic feasible solution, which means obtaining a new basis. The simplex method moves from one basic feasible solution to another by selecting a nonbasic variable to replace one of the current basic variables. This process of moving from one basic feasible solution to another is called an **iteration.** We now summarize the rules for selecting a nonbasic variable to be made basic and for selecting a current basic variable to be made nonbasic.

Criterion for Entering a New Variable into the Basis

Look at the net evaluation row ($c_j - z_j$), and select the variable to enter the basis that will cause the largest per-unit improvement in the value of the objective function. In the case of a tie, follow the convention of selecting the variable to enter the basis that corresponds to the leftmost of the columns.

To determine which basic variable will become nonbasic, only the positive coefficients in the incoming column correspond to basic variables that will decrease in value when the new basic variable enters.

Criterion for Removing a Variable from the Current Basis (Minimum Ratio Test)

Suppose the incoming basic variable corresponds to column j in the A portion of the simplex tableau. For each row i, compute the ratio b_i/a_{ij} for each a_{ij} greater than zero. The basic variable that will be removed from the basis corresponds to the minimum of these ratios. In case of a tie, we follow the convention of selecting the variable that corresponds to the uppermost of the tied rows.

To illustrate the computations involved, we add an extra column to the right of the tableau showing the b_i/a_{ij} ratios.

		x_1	x_2	s_1	s_2	s_3		$\frac{b_i}{a_{i1}}$
Basis	c_B	50	40	0	0	0		
s_1	0	3	5	1	0	0	**150**	$\frac{150}{3} = 50$
s_2	0	0	1	0	1	0	**20**	—
s_3	0	⑧	5	0	0	1	**300**	$\frac{300}{8} = 37.5$
z_j		0	0	0	0	0	**0**	
$c_j - z_j$		50	40	0	0	0		

We see that $c_1 - z_1 = 50$ is the largest positive value in the $c_j - z_j$ row. Hence, x_1 is selected to become the new basic variable. Checking the ratios b_i/a_{i1} for values of a_{i1} greater than zero, we see that $b_3/a_{31} = 300/8 = 37.5$ is the minimum of these ratios. Thus, the current basic variable associated with row 3 (s_3) is the variable selected to leave the basis. In the tableau we have circled $a_{31} = 8$ to indicate that the variable corresponding to the first column is to enter the basis and that the basic variable corresponding to the third row is to leave the basis. Adopting the usual linear programming terminology, we refer to this circled element as the **pivot element.** The column and the row containing the pivot element are called the **pivot column** and the **pivot row,** respectively.

The circled value is the pivot element; the corresponding column and row are called the pivot column and pivot row.

To improve the current solution of $x_1 = 0, x_2 = 0, s_1 = 150, s_2 = 20$, and $s_3 = 300$, we should increase x_1 to 37.5. The production of 37.5 units of the Deskpro results in a profit of $50(37.5) = 1875$. In producing 37.5 units of the Deskpro, s_3 will be reduced to zero. Hence, x_1 will become the new basic variable, replacing s_3 in the previous basis.

17.5 CALCULATING THE NEXT TABLEAU

We now want to update the simplex tableau in such a fashion that the column associated with the new basic variable is a unit column; in this way its value will be given by the right-hand-side value of the corresponding row. We would like the column in the new tableau corresponding to x_1 to look just like the column corresponding to s_3 in the original tableau, so our goal is to make the column in the A matrix corresponding to x_1 appear as

$$\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$$

The way in which we transform the simplex tableau so that it still represents an equivalent system of constraint equations is to use the following **elementary row operations.**

Elementary Row Operations

1. Multiply any row (equation) by a nonzero number.
2. Replace any row (equation) by the result of adding or subtracting a multiple of another row (equation) to it.

The application of these elementary row operations to a system of simultaneous linear equations will not change the solution to the system of equations; however, the elementary row operations will change the coefficients of the variables and the values of the right-hand sides.

The objective in performing elementary row operations is to transform the system of constraint equations into a form that makes it easy to identify the new basic feasible solution. Consequently, we must perform the elementary row operations in such a manner that we transform the column for the variable entering the basis into a unit column. We emphasize that the feasible solutions to the original constraint equations are the same as the feasible solutions to the modified constraint equations obtained by performing elementary row operations. However, many of the numerical values in the simplex tableau will change as the result of performing these row operations. Thus, the present method of referring to elements in the simplex tableau may lead to confusion.

Until now we made no distinction between the A matrix and b column coefficients in the tableau form of the problem and the corresponding coefficients in the simplex tableau. Indeed, we showed that the initial simplex tableau is formed by properly placing the a_{ij}, c_j, and b_i elements as given in the tableau form of the problem into the simplex tableau. To avoid confusion in subsequent simplex tableaus, we will refer to the portion of the simplex tableau that initially contained the a_{ij} values with the symbol $\bar{A}$, and the portion of the tableau that initially contained the b_i values with the symbol $\bar{b}$. In terms of the simplex tableau, elements in $\bar{A}$ will be denoted by $\bar{a}_{ij}$, and elements in $\bar{b}$ will be denoted by $\bar{b}_i$. In subsequent simplex tableaus, elementary row operations will change the tableau elements. The overbar notation should avoid any confusion when we wish to distinguish between (1) the original constraint coefficient values a_{ij} and right-hand-side values b_i of the tableau form, and (2) the simplex tableau elements $\bar{a}_{ij}$ and $\bar{b}_i$.

Now let us see how elementary row operations are used to create the next simplex tableau for the HighTech problem. Recall that the goal is to transform the column in the $\bar{A}$ portion of the simplex tableau corresponding to x_1 to a unit column; that is,

$$\begin{aligned} \bar{a}_{11} &= 0 \\ \bar{a}_{21} &= 0 \\ \bar{a}_{31} &= 1 \end{aligned}$$

To set $\bar{a}_{31} = 1$, we perform the first elementary row operation by multiplying the pivot row (row 3) by $\frac{1}{8}$ to obtain the equivalent equation

$$\tfrac{1}{8}(8x_1 + 5x_2 + 0s_1 + 0s_2 + 1s_3) = \tfrac{1}{8}(300)$$

or

$$1x_1 + \tfrac{5}{8}x_2 + 0s_1 + 0s_2 + \tfrac{1}{8}s_3 = \tfrac{75}{2} \qquad (17.9)$$

We refer to equation (17.9) in the updated simplex tableau as the *new pivot row.*

To set $\bar{a}_{11} = 0$, we perform the second elementary row operation by first multiplying the new pivot row by 3 to obtain the equivalent equation

$$3(1x_1 + \tfrac{5}{8}x_2 + 0s_1 + 0s_2 + \tfrac{1}{8}s_3) = 3(\tfrac{75}{2})$$

or

$$3x_1 + \tfrac{15}{8}x_2 + 0s_1 + 0s_2 + \tfrac{3}{8}s_3 = \tfrac{225}{2} \tag{17.10}$$

Subtracting equation (17.10) from the equation represented by row 1 of the simplex tableau completes the application of the second elementary row operation; thus, after dropping the terms with zero coefficients, we obtain

$$(3x_1 + 5x_2 + 1s_1) - (3x_1 + \tfrac{15}{8}x_2 + \tfrac{3}{8}s_3) = 150 - \tfrac{225}{2}$$

or

$$0x_1 + \tfrac{25}{8}x_2 + 1s_1 - \tfrac{3}{8}s_3 = \tfrac{75}{2} \tag{17.11}$$

Because $\bar{a}_{21} = 0$, no row operations need be performed on the second row of the simplex tableau. Replacing rows 1 and 3 with the coefficients in equations (17.11) and (17.9), respectively, we obtain the new simplex tableau

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
s_1	0	0	$\frac{25}{8}$	1	0	$-\frac{3}{8}$	$\frac{75}{2}$
s_2	0	0	1	0	1	0	**20**
x_1	50	1	$\frac{5}{8}$	0	0	$\frac{1}{8}$	$\frac{75}{2}$
	z_j						**1875**
	$c_j - z_j$						

Assigning zero values to the nonbasic variables x_2 and s_3 permits us to identify the following new basic feasible solution:

$$s_1 = \tfrac{75}{2}$$
$$s_2 = 20$$
$$x_1 = \tfrac{75}{2}$$

This solution is also provided by the last column in the new simplex tableau. The profit associated with this solution is obtained by multiplying the solution values for the basic variables as given in the $\bar{b}$ column by their corresponding objective function coefficients as given in the c_B column; that is,

$$0(\tfrac{75}{2}) + 0(20) + 50(\tfrac{75}{2}) = 1875$$

Interpreting the Results of an Iteration

In our example, the initial basic feasible solution was

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ s_1 &= 150 \\ s_2 &= 20 \\ s_3 &= 300 \end{aligned}$$

with a corresponding profit of \$0. One iteration of the simplex method moved us to another basic feasible solution with an objective function value of \$1875. This new basic feasible solution is

$$\begin{aligned} x_1 &= 75/2 \\ x_2 &= 0 \\ s_1 &= 75/2 \\ s_2 &= 20 \\ s_3 &= 0 \end{aligned}$$

In Figure 17.2 we see that the initial basic feasible solution corresponds to extreme point ①. The first iteration moved us in the direction of the greatest increase per unit in profit—that is, along the x_1 axis. We moved away from extreme point ① in the x_1 direction until we could not move farther without violating one of the constraints. The tableau we obtained after one iteration provides the basic feasible solution corresponding to extreme point ②.

The first iteration moves us from the origin in Figure 17.2 to extreme point 2.

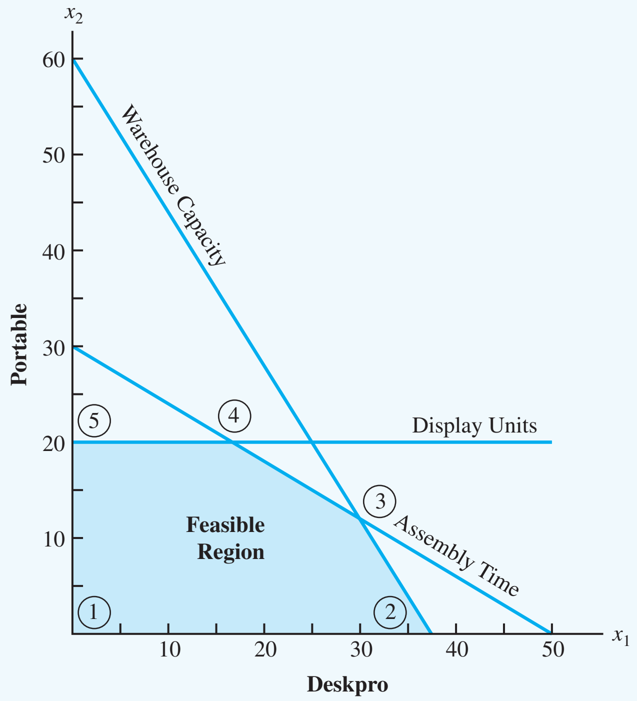
We note from Figure 17.2 that at extreme point ② the warehouse capacity constraint is binding with $s_3 = 0$ and that the other two constraints contain slack. From the simplex tableau, we see that the amount of slack for these two constraints is given by $s_1 = 75/2$ and $s_2 = 20$.

Moving Toward a Better Solution

To see whether a better basic feasible solution can be found, we need to calculate the z_j and $c_j - z_j$ rows for the new simplex tableau. Recall that the elements in the z_j row are the sum of the products obtained by multiplying the elements in the c_B column of the simplex tableau by the corresponding elements in the columns of the $\bar{A}$ matrix. Thus, we obtain

$$\begin{aligned} z_1 &= 0(0) + 0(0) + 50(1) = 50 \\ z_2 &= 0(25/8) + 0(1) + 50(5/8) = 250/8 \\ z_3 &= 0(1) + 0(0) + 50(0) = 0 \\ z_4 &= 0(0) + 0(1) + 50(0) = 0 \\ z_5 &= 0(-3/8) + 0(0) + 50(1/8) = 50/8 \end{aligned}$$

FIGURE 17.2 FEASIBLE REGION AND EXTREME POINTS FOR THE HIGHTECH INDUSTRIES PROBLEM



Subtracting z_j from c_j to compute the new net evaluation row, we obtain the following simplex tableau:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
s_1	0	0	$\frac{25}{8}$	1	0	$-\frac{3}{8}$	**$\frac{75}{2}$**
s_2	0	0	1	0	1	0	**20**
x_1	50	1	$\frac{5}{8}$	0	0	$\frac{1}{8}$	**$\frac{75}{2}$**
z_j		50	$\frac{250}{8}$	0	0	$\frac{50}{8}$	**1875**
$c_j - z_j$		0	$\frac{70}{8}$	0	0	$-\frac{50}{8}$	

Let us now analyze the $c_j - z_j$ row to see whether we can introduce a new variable into the basis and continue to improve the value of the objective function. Using the rule for determining which variable should enter the basis next, we select x_2 because it has the highest positive coefficient in the $c_j - z_j$ row.

To determine which variable will be removed from the basis when x_2 enters, we must compute for each row i the ratio $\bar{b}_i/\bar{a}_{i2}$ (remember, though, that we should compute this ratio only if $\bar{a}_{i2}$ is greater than zero); then we select the variable to leave the basis that corresponds to the minimum ratio. As before, we will show these ratios in an extra column of the simplex tableau:

		x_1	x_2	s_1	s_2	s_3		$\frac{\bar{b}_i}{\bar{a}_{i2}}$
Basis	c_B	50	40	0	0	0		
s_1	0	0	$\enclose{circle}{25/8}$	1	0	$-3/8$	**$75/2$**	$\frac{75/2}{25/8} = 12$
s_2	0	0	1	0	1	0	**20**	$\frac{20}{1} = 20$
x_1	50	1	$5/8$	0	0	$1/8$	**$75/2$**	$\frac{75/2}{5/8} = 60$
z_j		50	$250/8$	0	0	$50/8$	**1875**	
$c_j - z_j$		0	$70/8$	0	0	$-50/8$		

With 12 as the minimum ratio, s_1 will leave the basis. The pivot element is $\bar{a}_{12} = 25/8$, which is circled in the preceding tableau. The nonbasic variable x_2 must now be made a basic variable in row 1. This requirement means that we must perform the elementary row operations that will convert the x_2 column into a unit column with a 1 in row 1; that is, we will have to transform the second column in the tableau to the form

$$\begin{array}{c} 1 \\ 0 \\ 0 \end{array}$$

We can make this change by performing the following elementary row operations:

Step 1. Multiply every element in row 1 (the pivot row) by $8/25$ in order to make $\bar{a}_{12} = 1$.
Step 2. Subtract the new row 1 (the new pivot row) from row 2 to make $\bar{a}_{22} = 0$.
Step 3. Multiply the new pivot row by $5/8$, and subtract the result from row 3 to make $\bar{a}_{32} = 0$.

The new simplex tableau resulting from these row operations is as follows:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
x_2	40	0	1	$8/25$	0	$-3/25$	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	**8**
x_1	50	1	0	$-5/25$	0	$5/25$	**30**
z_j		50	40	$14/5$	0	$26/5$	**1980**
$c_j - z_j$		0	0	$-14/5$	0	$-26/5$	

Note that the values of the basic variables are $x_2 = 12$, $s_2 = 8$, and $x_1 = 30$, and the corresponding profit is $40(12) + 0(8) + 50(30) = 1980$.

We must now determine whether to bring any other variable into the basis and thereby move to another basic feasible solution. Looking at the net evaluation row, we see that every element is zero or negative. Because $c_j - z_j$ is less than or equal to zero for both of the nonbasic variables s_1 and s_3, any attempt to bring a nonbasic variable into the basis at this point will result in a lowering of the current value of the objective function. Hence, this tableau represents the optimal solution. In general, the simplex method uses the following criterion to determine when the optimal solution has been obtained.

Optimality Criterion

The optimal solution to a linear programming problem has been reached when all of the entries in the net evaluation row $(c_j - z_j)$ are zero or negative. In such cases, the optimal solution is the current basic feasible solution.

Referring to Figure 17.2, we can see graphically the process that the simplex method used to determine an optimal solution. The initial basic feasible solution corresponds to the origin ($x_1 = 0$, $x_2 = 0$, $s_1 = 150$, $s_2 = 20$, $s_3 = 300$). The first iteration caused x_1 to enter the basis and s_3 to leave. The second basic feasible solution corresponds to extreme point ② ($x_1 = 75/2$, $x_2 = 0$, $s_1 = 75/2$, $s_2 = 20$, $s_3 = 0$). At the next iteration, x_2 entered the basis and s_1 left. This iteration brought us to extreme point ③ and the optimal solution ($x_1 = 30$, $x_2 = 12$, $s_1 = 0$, $s_2 = 8$, $s_3 = 0$).

For the HighTech problem with only two decision variables, we had a choice of using the graphical or simplex method. For problems with more than two variables, we shall always use the simplex method.

Interpreting the Optimal Solution

Using the final simplex tableau, we find the optimal solution to the HighTech problem consists of the basic variables x_1, x_2, and s_2 and nonbasic variables s_1 and s_3 with:

$$\begin{aligned} x_1 &= 30 \\ x_2 &= 12 \\ s_1 &= 0 \\ s_2 &= 8 \\ s_3 &= 0 \end{aligned}$$

The value of the objective function is $1980. If management wants to maximize the total profit contribution, HighTech should produce 30 units of the Deskpro and 12 units of the Portable. When $s_2 = 8$, management should note that there will be eight unused Portable display units. Moreover, because $s_1 = 0$ and $s_3 = 0$, no slack is associated with the assembly time constraint and the warehouse capacity constraint; in other words, these constraints are both binding. Consequently, if it is possible to obtain additional assembly time and/or additional warehouse space, management should consider doing so.

Figure 17.3 shows the computer solution to the HighTech problem using The Management Scientist software package. The optimal solution with $x_1 = 30$ and $x_2 = 12$ is shown to have an objective function value of $1980. The values of the slack variables complete the optimal solution with $s_1 = 0$, $s_2 = 8$, and $s_3 = 0$. The values in the Reduced Costs column are from the net evaluation row of the final simplex tableau. Note that the $c_j - z_j$ values in columns corresponding to x_1 and x_2 are both 0. The dual prices are the z_j values for the three slack variables in the final simplex tableau. Referring to the final tableau, we see that the dual price for constraint 1 is the z_j value corresponding to s_1 where $14/5 = 2.8$. Similarly, the dual price for constraint 2 is 0, and the dual price for constraint 3 is $26/5 = 5.2$. The use of the simplex method to compute dual prices will be discussed further when we cover sensitivity analysis in Chapter 18.

FIGURE 17.3 THE MANAGEMENT SCIENTIST SOLUTION FOR THE HIGHTECH INDUSTRIES PROBLEM

```
OPTIMAL SOLUTION

Objective Function Value =           1980.000

      Variable              Value            Reduced Costs
   --------------     ---------------     -----------------
         X1                  30.000                 0.000
         X2                  12.000                 0.000

     Constraint          Slack/Surplus        Dual Prices
   --------------     ---------------     -----------------
        1                     0.000                 2.800
        2                     8.000                 0.000
        3                     0.000                 5.200
```

Summary of the Simplex Method

Let us now summarize the steps followed to solve a linear program using the simplex method. We assume that the problem has all less-than-or-equal-to constraints and involves maximization.

Step 1. Formulate a linear programming model of the problem.

Step 2. Add slack variables to each constraint to obtain standard form. This also provides the tableau form necessary to identify an initial basic feasible solution for problems involving all less-than-or-equal-to constraints with nonnegative right-hand-side values.

Step 3. Set up the initial simplex tableau.

Step 4. Choose the nonbasic variable with the largest entry in the net evaluation row to bring into the basis. This variable identifies the pivot column: the column associated with the incoming variable.

Step 5. Choose as the pivot row that row with the smallest ratio of $\bar{b}_i/\bar{a}_{ij}$ for $\bar{a}_{ij} > 0$ where j is the pivot column. This pivot row is the row of the variable leaving the basis when variable j enters.

Step 6. Perform the necessary elementary row operations to convert the column for the incoming variable to a unit column with a 1 in the pivot row.

   a. Divide each element of the pivot row by the pivot element (the element in the pivot row and pivot column).

   b. Obtain zeroes in all other positions of the pivot column by adding or subtracting an appropriate multiple of the new pivot row. Once the row operations have been completed, the value of the new basic feasible solution can be read from the $\bar{b}$ column of the tableau.

Step 7. Test for optimality. If $c_j - z_j \leq 0$ for all columns, the solution is optimal. If not, return to step 4.

The steps are basically the same for problems with equality and greater-than-or-equal-to constraints except that setting up tableau form requires a little more work. We discuss what is involved in Section 17.6. The modification necessary for minimization problems is covered in Section 17.7.

To test your ability to solve a problem employing the simplex method, try Problem 6.

NOTES AND COMMENTS

The entries in the net evaluation row provide the reduced costs that appear in the computer solution to a linear program. Recall that in Chapter 3 we defined the reduced cost as the amount by which an objective function coefficient would have to improve before it would be possible for the corresponding variable to assume a positive value in the optimal solution. In general, the reduced costs are the absolute values of the entries in the net evaluation row.

17.6 TABLEAU FORM: THE GENERAL CASE

This section explains how to get started with the simplex method for problems with greater-than-or-equal-to and equality constraints.

When a linear program contains all less-than-or-equal-to constraints with nonnegative right-hand-side values, it is easy to set up the tableau form; we simply add a slack variable to each constraint. However, obtaining tableau form is somewhat more complex if the linear program contains greater-than-or-equal-to constraints, equality constraints, and/or negative right-hand-side values. In this section we describe how to develop tableau form for each of these situations and also how to solve linear programs involving equality and greater-than-or-equal-to constraints using the simplex method.

Greater-Than-or-Equal-to Constraints

Suppose that in the HighTech Industries problem, management wanted to ensure that the combined total production for both models would be at least 25 units. This requirement means that the following constraint must be added to the current linear program:

$$1x_1 + 1x_2 \geq 25$$

Adding this constraint results in the following modified problem:

$$\begin{array}{lrcll}
\text{Max} & 50x_1 + 40x_2 & & & \\
\text{s.t.} & & & & \\
& 3x_1 + 5x_2 & \leq & 150 & \text{Assembly time} \\
& 1x_2 & \leq & 20 & \text{Portable display} \\
& 8x_1 + 5x_2 & \leq & 300 & \text{Warehouse space} \\
& 1x_1 + 1x_2 & \geq & 25 & \text{Minimum total production} \\
& x_1, x_2 \geq 0 & & &
\end{array}$$

First, we use three slack variables and one surplus variable to write the problem in standard form. This provides the following:

$$\begin{array}{ll}
\text{Max} & 50x_1 + 40x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\
\text{s.t.} & \\
& 3x_1 + 5x_2 + 1s_1 = 150 \qquad (17.12) \\
& 1x_2 + 1s_2 = 20 \qquad (17.13) \\
& 8x_1 + 5x_2 + 1s_3 = 300 \qquad (17.14) \\
& 1x_1 + 1x_2 - 1s_4 = 25 \qquad (17.15) \\
& x_1, x_2, s_1, s_2, s_3, s_4 \geq 0
\end{array}$$

Now let us consider how we obtain an initial basic feasible solution to start the simplex method. Previously, we set $x_1 = 0$ and $x_2 = 0$ and selected the slack variables as the initial basic variables. The extension of this notion to the modified HighTech problem would suggest setting $x_1 = 0$ and $x_2 = 0$ and selecting the slack and surplus variables as the initial basic variables. Doing so results in the basic solution

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ s_1 &= 150 \\ s_2 &= 20 \\ s_3 &= 300 \\ s_4 &= -25 \end{aligned}$$

Clearly this solution is not a basic feasible solution because $s_4 = -25$ violates the nonnegativity requirement. The difficulty is that the standard form and the tableau form are not equivalent when the problem contains greater-than-or-equal-to constraints.

To set up the tableau form, we shall resort to a mathematical "trick" that will enable us to find an initial basic feasible solution in terms of the slack variables s_1, s_2, and s_3 and a new variable we shall denote a_4. The new variable constitutes the mathematical trick. Variable a_4 really has nothing to do with the HighTech problem; it merely enables us to set up the tableau form and thus obtain an initial basic feasible solution. This new variable, which has been artificially created to start the simplex method, is referred to as an **artificial variable.**

Artificial variables are appropriately named; they have no physical meaning in the real problem.

The notation for artificial variables is similar to the notation used to refer to the elements of the *A* matrix. To avoid any confusion between the two, recall that the elements of the *A* matrix (constraint coefficients) always have two subscripts, whereas artificial variables have only one subscript.

With the addition of an artificial variable, we can convert the standard form of the problem into tableau form. We add artificial variable a_4 to constraint equation (17.15) to obtain the following representation of the system of equations in tableau form:

$$\begin{array}{lllllll} 3x_1 + 5x_2 + 1s_1 & & & & & = & 150 \\ \phantom{3x_1 + {}} 1x_2 & + 1s_2 & & & & = & 20 \\ 8x_1 + 5x_2 & & + 1s_3 & & & = & 300 \\ 1x_1 + 1x_2 & & & - 1s_4 & + 1a_4 & = & 25 \end{array}$$

Note that the subscript on the artificial variable identifies the constraint with which it is associated. Thus, a_4 is the artificial variable associated with the fourth constraint.

Because the variables s_1, s_2, s_3, and a_4 each appear in a different constraint with a coefficient of 1, and the right-hand-side values are nonnegative, both requirements of the tableau form have been satisfied. We can now obtain an initial basic feasible solution by setting $x_1 = x_2 = s_4 = 0$. The complete solution is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ s_1 &= 150 \\ s_2 &= 20 \\ s_3 &= 300 \\ s_4 &= 0 \\ a_4 &= 25 \end{aligned}$$

Is this solution feasible in terms of the real HighTech problem? No, it is not. It does not satisfy the constraint 4 combined total production requirement of 25 units. We must make an important distinction between a basic feasible solution for the tableau form and a feasible solution for the real problem. A basic feasible solution for the tableau form of a linear programming problem is not always a feasible solution for the real problem.

A basic feasible solution containing one or more artificial variables at positive values is not feasible for the real problem.

The reason for creating the tableau form is to obtain the initial basic feasible solution that is required to start the simplex method. Thus, we see that whenever it is necessary to introduce artificial variables, the initial simplex solution will not in general be feasible for the real problem. This situation is not as difficult as it might seem, however, because the only time we must have a feasible solution for the real problem is at the last iteration of the simplex method. Thus, devising a way to guarantee that any artificial variable would be eliminated from the basic feasible solution before the optimal solution is reached would eliminate the difficulty.

The way in which we guarantee that artificial variables will be eliminated before the optimal solution is reached is to assign each artificial variable a very large cost in the objective function. For example, in the modified HighTech problem, we could assign a very large negative number as the profit coefficient for artificial variable a_4. Hence, if this variable is in the basis, it will substantially reduce profits. As a result, this variable will be eliminated from the basis as soon as possible, which is precisely what we want to happen.

As an alternative to picking a large negative number such as $-100{,}000$ for the profit coefficient, we will denote the profit coefficient of each artificial variable by $-M$. Here it is assumed that M represents a very large number—in other words, a number of large magnitude and hence, the letter M. This notation will make it easier to keep track of the elements of the simplex tableau that depend on the profit coefficients of the artificial variables. Using $-M$ as the profit coefficient for artificial variable a_4 in the modified HighTech problem, we can write the objective function for the tableau form of the problem as follows:

$$\text{Max} \quad 50x_1 + 40x_2 + 0s_1 + 0s_2 + 0s_3 + 0s_4 - Ma_4$$

The initial simplex tableau for the problem is shown here.

		x_1	x_2	s_1	s_2	s_3	s_4	a_4	
Basis	c_B	50	40	0	0	0	0	$-M$	
s_1	0	3	5	1	0	0	0	0	**150**
s_2	0	0	1	0	1	0	0	0	**20**
s_3	0	8	5	0	0	1	0	0	**300**
a_4	$-M$	①	1	0	0	0	-1	1	**25**
z_j		$-M$	$-M$	0	0	0	M	$-M$	**$-25M$**
$c_j - z_j$		$50 + M$	$40 + M$	0	0	0	$-M$	0	

This tableau corresponds to the solution $s_1 = 150$, $s_2 = 20$, $s_3 = 300$, $a_4 = 25$, and $x_1 = x_2 = s_4 = 0$. In terms of the simplex tableau, this solution is a basic feasible solution because all the variables are greater than or equal to zero, and $n - m = 7 - 4 = 3$ of the variables are equal to zero.

Because $c_1 - z_1 = 50 + M$ is the largest value in the net evaluation row, we see that x_1 will become a basic variable during the first iteration of the simplex method. Further calculations with the simplex method show that x_1 will replace a_4 in the basic solution. The following simplex tableau is the result of the first iteration.

Result of Iteration 1

		x_1	x_2	s_1	s_2	s_3	s_4	a_4	
Basis	c_B	50	40	0	0	0	0	$-M$	
s_1	0	0	2	1	0	0	3	-3	**75**
s_2	0	0	1	0	1	0	0	0	**20**
s_3	0	0	-3	0	0	1	8	-8	**100**
x_1	50	1	1	0	0	0	-1	1	**25**
z_j		50	50	0	0	0	-50	50	**1250**
$c_j - z_j$		0	-10	0	0	0	50	$-M - 50$	

When the artificial variable $a_4 = 0$, we have a situation in which the basic feasible solution contained in the simplex tableau is also a feasible solution to the real HighTech problem. In addition, because a_4 is an artificial variable that was added simply to obtain an initial basic feasible solution, we can now drop its associated column from the simplex tableau. Indeed, whenever artificial variables are used, they can be dropped from the simplex tableau as soon as they have been eliminated from the basic feasible solution.

When artificial variables are required to obtain an initial basic feasible solution, the iterations required to eliminate the artificial variables are referred to as **phase I** of the simplex method. When all the artificial variables have been eliminated from the basis, phase I is complete, and a basic feasible solution to the real problem has been obtained. Thus, by dropping the column associated with a_4 from the current tableau, we obtain the following simplex tableau at the end of phase I.

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_B	50	40	0	0	0	0	
s_1	0	0	2	1	0	0	3	**75**
s_2	0	0	1	0	1	0	0	**20**
s_3	0	0	-3	0	0	1	⑧	**100**
x_1	50	1	1	0	0	0	-1	**25**
z_j		50	50	0	0	0	-50	**1250**
$c_j - z_j$		0	-10	0	0	0	50	

We are now ready to begin phase II of the simplex method. This phase simply continues the simplex method computations after all artificial variables have been removed. At the next iteration, variable s_4 with $c_j - z_j = 50$ is entered into the solution and variable s_3 is eliminated. The simplex tableau after this iteration is:

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_B	50	40	0	0	0	0	
s_1	0	0	$\textcircled{25/8}$	1	0	$-3/8$	0	**75/2**
s_2	0	0	1	0	1	0	0	**20**
s_4	0	0	$-3/8$	0	0	$1/8$	1	**25/2**
x_1	50	1	$5/8$	0	0	$1/8$	0	**75/2**
z_j		50	$250/8$	0	0	$50/8$	0	**1875**
$c_j - z_j$		0	$70/8$	0	0	$-50/8$	0	

One more iteration is required. This time x_2 comes into the solution, and s_1 is eliminated. After performing this iteration, the following simplex tableau shows that the optimal solution has been reached.

		x_1	x_2	s_1	s_2	s_3	s_4	
Basis	c_B	50	40	0	0	0	0	
x_2	40	0	1	$8/25$	0	$-3/25$	0	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	0	**8**
s_4	0	0	0	$3/25$	0	$2/25$	1	**17**
x_1	50	1	0	$-5/25$	0	$5/25$	0	**30**
z_j		50	40	$14/5$	0	$26/5$	0	**1980**
$c_j - z_j$		0	0	$-14/5$	0	$-26/5$	0	

It turns out that the optimal solution to the modified HighTech problem is the same as the solution for the original problem. However, the simplex method required more iterations to reach this extreme point, because an extra iteration was needed to eliminate the artificial variable (a_4) in phase I.

Fortunately, once we obtain an initial simplex tableau using artificial variables, we need not concern ourselves with whether the basic solution at a particular iteration is feasible for the real problem. We need only follow the rules for the simplex method. If we reach the optimality criterion (all $c_j - z_j \leq 0$) and all the artificial variables have been eliminated from the solution, then we have found the optimal solution. On the other hand, if we reach the optimality criterion and one or more of the artificial variables remain in solution at a positive value, then there is no feasible solution to the problem. This special case will be discussed further in Section 17.8.

Equality Constraints

When an equality constraint occurs in a linear programming problem, we need to add an artificial variable to obtain tableau form and an initial basic feasible solution. For example, if constraint 1 is

$$6x_1 + 4x_2 - 5x_3 = 30$$

we would simply add an artificial variable a_1 to create a basic feasible solution in the initial simplex tableau. With the artificial variable, the constraint equation becomes

$$6x_1 + 4x_2 - 5x_3 + 1a_1 = 30$$

Now a_1 can be selected as the basic variable for this row, and its value is given by the right-hand side. Once we have created tableau form by adding an artificial variable to each equality constraint, the simplex method proceeds exactly as before.

Eliminating Negative Right-Hand-Side Values

One of the properties of the tableau form of a linear program is that the values on the right-hand sides of the constraints have to be nonnegative. In formulating a linear programming problem, we may find one or more of the constraints have negative right-hand-side values. To see how this situation might happen, suppose that the management of HighTech has specified that the number of units of the Portable model, x_2, has to be less than or equal to the number of units of the Deskpro model, x_1, after setting aside five units of the Deskpro for internal company use. We could formulate this constraint as

$$x_2 \leq x_1 - 5 \tag{17.16}$$

Subtracting x_1 from both sides of the inequality places both variables on the left-hand side of the inequality. Thus,

$$-x_1 + x_2 \leq -5 \tag{17.17}$$

Because this constraint has a negative right-hand-side value, we can develop an equivalent constraint with a nonnegative right-hand-side value by multiplying both sides of the constraint by -1. In doing so, we recognize that multiplying an inequality constraint by -1 changes the direction of the inequality.

Thus, to convert inequality (17.17) to an equivalent constraint with a nonnegative right-hand-side value, we multiply by -1 to obtain

$$x_1 - x_2 \geq 5 \tag{17.18}$$

We now have an acceptable nonnegative right-hand-side value. Tableau form for this constraint can now be obtained by subtracting a surplus variable and adding an artificial variable.

For a greater-than-or-equal-to constraint, multiplying by -1 creates an equivalent less-than-or-equal-to constraint. For example, suppose we had the following greater-than-or-equal-to constraint:

$$6x_1 + 3x_2 - 4x_3 \geq -20$$

Multiplying by -1 to obtain an equivalent constraint with a nonnegative right-hand-side value leads to the following less-than-or-equal-to constraint

$$-6x_1 - 3x_2 + 4x_3 \leq 20$$

Tableau form can be created for this constraint by adding a slack variable.

For an equality constraint with a negative right-hand-side value, we simply multiply by -1 to obtain an equivalent constraint with a nonnegative right-hand-side value. An artificial variable can then be added to create the tableau form.

Summary of the Steps to Create Tableau Form

Step 1. If the original formulation of the linear programming problem contains one or more constraints with negative right-hand-side values, multiply each of these constraints by -1. Multiplying by -1 will change the direction of the inequalities. This step will provide an equivalent linear program with nonnegative right-hand-side values.

Step 2. For $\leq$ constraints, add a slack variable to obtain an equality constraint. The coefficient of the slack variable in the objective function is assigned a value of zero. It provides the tableau form for the constraint, and the slack variable becomes one of the basic variables in the initial basic feasible solution.

Step 3. For $\geq$ constraints, subtract a surplus variable to obtain an equality constraint, and then add an artificial variable to obtain the tableau form. The coefficient of the surplus variable in the objective function is assigned a value of zero. The coefficient of the artificial variable in the objective function is assigned a value of $-M$. The artificial variable becomes one of the basic variables in the initial basic feasible solution.

Step 4. For equality constraints, add an artificial variable to obtain the tableau form. The coefficient of the artificial variable in the objective function is assigned a value of $-M$. The artificial variable becomes one of the basic variables in the initial basic feasible solution.

To obtain some practice in applying these steps, convert the following example problem into tableau form, and then set up the initial simplex tableau:

$$
\begin{array}{ll}
\text{Max} & 6x_1 + 3x_2 + 4x_3 + 1x_4 \\
\text{s.t.} & \\
& -2x_1 - \tfrac{1}{2}x_2 + 1x_3 - 6x_4 = -60 \\
& 1x_1 + 1x_3 + \tfrac{2}{3}x_4 \leq 20 \\
& -1x_2 - 5x_3 \leq -50 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

To eliminate the negative right-hand-side values in constraints 1 and 3, we apply step 1. Multiplying both constraints by -1, we obtain the following equivalent linear program:

$$
\begin{array}{ll}
\text{Max} & 6x_1 + 3x_2 + 4x_3 + 1x_4 \\
\text{s.t.} & \\
& 2x_1 + \tfrac{1}{2}x_2 - 1x_3 + 6x_4 = 60 \\
& 1x_1 + 1x_3 + \tfrac{2}{3}x_4 \leq 20 \\
& 1x_2 + 5x_3 \geq 50 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{array}
$$

Note that the direction of the $\leq$ inequality in constraint 3 has been reversed as a result of multiplying the constraint by -1. By applying step 4 for constraint 1, step 2 for constraint 2, and step 3 for constraint 3, we obtain the following tableau form:

$$\begin{array}{lrcrcrcrcrcrcrcr}
\text{Max} & 6x_1 & + & 3x_2 & + & 4x_3 & + & 1x_4 & + & 0s_2 & + & 0s_3 & - & Ma_1 & - & Ma_3 \\
\text{s.t.} & & & & & & & & & & & & & & & \\
& 2x_1 & + & \frac{1}{2}x_2 & - & 1x_3 & + & 6x_4 & & & & & + & 1a_1 & & = 60 \\
& 1x_1 & & & + & 1x_3 & + & \frac{2}{3}x_4 & + & 1s_2 & & & & & & = 20 \\
& & & 1x_2 & + & 5x_3 & & & & & - & 1s_3 & & & + & 1a_3 = 50 \\
& \multicolumn{15}{l}{x_1, x_2, x_3, x_4, s_2, s_3, a_1, a_3 \geq 0}
\end{array}$$

The initial simplex tableau corresponding to this tableau form is

		x_1	x_2	x_3	x_4	s_2	s_3	a_1	a_3	
Basis	c_B	6	3	4	1	0	0	$-M$	$-M$	
a_1	$-M$	2	$\frac{1}{2}$	-1	$\textcircled{6}$	0	0	1	0	**60**
s_2	0	1	0	1	$\frac{2}{3}$	1	0	0	0	**20**
a_3	$-M$	0	1	5	0	0	-1	0	1	**50**
z_j		$-2M$	$-\frac{3}{2}M$	$-4M$	$-6M$	0	M	$-M$	$-M$	$\mathbf{-110M}$
$c_j - z_j$		$6 + 2M$	$3 + \frac{3}{2}M$	$4 + 4M$	$1 + 6M$	0	$-M$	0	0	

For practice setting up tableau form and developing the initial simplex tableau for problems with any constraint form, try Problem 15.

Note that we have circled the pivot element indicating that x_4 will enter and a_1 will leave the basis at the first iteration.

NOTES AND COMMENTS

We have shown how to convert constraints with negative right-hand sides to equivalent constraints with positive right-hand sides. Actually, nothing is wrong with formulating a linear program and including negative right-hand sides. But if you want to use the ordinary simplex method to solve the linear program, you must first alter the constraints to eliminate the negative right-hand sides.

17.7 SOLVING A MINIMIZATION PROBLEM

We can use the simplex method to solve a minimization problem in two ways. The first approach requires that we change the rule used to introduce a variable into the basis. Recall that in the maximization case, we select the variable with the largest positive $c_j - z_j$ as the variable to introduce next into the basis, because the value of $c_j - z_j$ tells us the amount the objective function will increase if one unit of the variable in column j is brought into solution. To solve the minimization problem, we simply reverse this rule. That is, we select the variable with the most negative $c_j - z_j$ as the one to introduce next. Of course, this approach means the stopping rule for the optimal solution will also have to be changed. Using this approach to solve a minimization problem, we would stop when every value in the net evaluation row is zero or positive.

The second approach to solving a minimization problem is the one we shall employ in this book. It is based on the fact that any minimization problem can be converted to an equivalent maximization problem by multiplying the objective function by -1. Solving the resulting maximization problem will provide the optimal solution to the minimization problem.

Let us illustrate this second approach by using the simplex method to solve the M&D Chemicals problem introduced in Chapter 2. Recall that in this problem, management wanted to minimize the cost of producing two products subject to a demand constraint for product A, a minimum total production quantity requirement, and a constraint on available processing time. The mathematical statement of the M&D Chemicals problem is shown here.

In keeping with the general notation of this chapter, we are using x_1 and x_2 to represent units of product A and product B.

$$
\begin{array}{lrcll}
\text{Min} & 2x_1 + 3x_2 & & & \\
\text{s.t.} & & & & \\
& 1x_1 & \geq & 125 & \text{Demand for product A} \\
& 1x_1 + 1x_2 & \geq & 350 & \text{Total production} \\
& 2x_1 + 1x_2 & \leq & 600 & \text{Processing time} \\
& x_1, x_2 & \geq & 0 &
\end{array}
$$

To solve this problem using the simplex method, we first multiply the objective function by -1 to convert the minimization problem into the following equivalent maximization problem:

We convert a minimization problem to a maximization problem by multiplying the objective function by -1.

$$
\begin{array}{lrcll}
\text{Max} & -2x_1 - 3x_2 & & & \\
\text{s.t.} & & & & \\
& 1x_1 & \geq & 125 & \text{Demand for product A} \\
& 1x_1 + 1x_2 & \geq & 350 & \text{Total production} \\
& 2x_1 + 1x_2 & \leq & 600 & \text{Processing time} \\
& x_1, x_2 & \geq & 0 &
\end{array}
$$

The tableau form for this problem is as follows:

$$
\begin{array}{ll}
\text{Max} & -2x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 - Ma_1 - Ma_2 \\
\text{s.t.} & \\
& 1x_1 \qquad\quad - 1s_1 \qquad\qquad\qquad\quad + 1a_1 \qquad\quad = 125 \\
& 1x_1 + 1x_2 \qquad - 1s_2 \qquad\qquad\qquad + 1a_2 = 350 \\
& 2x_1 + 1x_2 \qquad\qquad + 1s_3 \qquad\qquad\qquad = 600 \\
& x_1, x_2, s_1, s_2, s_3, a_1, a_2 \geq 0
\end{array}
$$

The initial simplex tableau is shown here:

		x_1	x_2	s_1	s_2	s_3	a_1	a_2	
Basis	c_B	-2	-3	0	0	0	$-M$	$-M$	
a_1	$-M$	①	0	-1	0	0	1	0	**125**
a_2	$-M$	1	1	0	-1	0	0	1	**350**
s_3	0	2	1	0	0	1	0	0	**600**
z_j		$-2M$	$-M$	M	M	0	$-M$	$-M$	**$-475M$**
$c_j - z_j$		$-2 + 2M$	$-3 + M$	$-M$	$-M$	0	0	0	

At the first iteration, x_1 is brought into the basis and a_1 is removed. After dropping the a_1 column from the tableau, the result of the first iteration is as follows:

		x_1	x_2	s_1	s_2	s_3	a_2	
Basis	c_B	-2	-3	0	0	0	$-M$	
x_1	-2	1	0	-1	0	0	0	**125**
a_2	$-M$	0	1	1	-1	0	1	**225**
s_3	0	0	1	②	0	1	0	**350**
z_j		-2	$-M$	$2-M$	M	0	$-M$	$\mathbf{-250-225M}$
$c_j - z_j$		0	$-3+M$	$-2+M$	$-M$	0	0	

Continuing with two more iterations of the simplex method provides the following final simplex tableau:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	-2	-3	0	0	0	
x_1	-2	1	0	0	1	1	**250**
x_2	-3	0	1	0	-2	-1	**100**
s_1	0	0	0	1	1	1	**125**
z_j		-2	-3	0	4	1	**−800**
$c_j - z_j$		0	0	0	-4	-1	

The value of the objective function -800 must be multiplied by -1 to obtain the value of the objective function for the original minimization problem. Thus, the minimum total cost of the optimal solution is \$800.

Try Problem 17 for practice solving a minimization problem with the simplex method.

In the next section we discuss some important special cases that may occur when trying to solve any linear programming problem. We will only consider the case for maximization problems, recognizing that all minimization problems can be converted into an equivalent maximization problem by multiplying the objective function by -1.

17.8 SPECIAL CASES

In Chapter 2 we discussed how infeasibility, unboundedness, and alternative optimal solutions could occur when solving linear programming problems using the graphical solution procedure. These special cases can also arise when using the simplex method. In addition, a special case referred to as *degeneracy* can theoretically cause difficulties for the simplex method. In this section we show how these special cases can be recognized and handled when the simplex method is used.

Infeasibility

Infeasibility occurs whenever no solution to the linear program can be found that satisfies all the constraints, including the nonnegativity constraints. Let us now see how infeasibility is recognized when the simplex method is used.

In Section 17.6, when discussing artificial variables, we mentioned that infeasibility can be recognized when the optimality criterion indicates that an optimal solution has been obtained and one or more of the artificial variables remain in the solution at a positive value. As an illustration of this situation, let us consider another modification of the HighTech Industries problem. Suppose management imposed a minimum combined total production requirement of 50 units. The revised problem formulation is shown as follows.

$$
\begin{array}{lrll}
\text{Max} & 50x_1 + 40x_2 & & \\
\text{s.t.} & & & \\
& 3x_1 + 5x_2 \le 150 & & \text{Assembly time} \\
& 1x_2 \le 20 & & \text{Portable display} \\
& 8x_1 + 5x_2 \le 300 & & \text{Warehouse space} \\
& 1x_1 + 1x_2 \ge 50 & & \text{Minimum total production} \\
& x_1, x_2 \ge 0 & &
\end{array}
$$

Two iterations of the simplex method will provide the following tableau:

		x_1	x_2	s_1	s_2	s_3	s_4	a_4	
Basis	c_B	50	40	0	0	0	0	$-M$	
x_2	40	0	1	$8/25$	0	$-3/25$	0	0	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	0	0	**8**
x_1	50	1	0	$-5/25$	0	$5/25$	0	0	**30**
a_4	$-M$	0	0	$-3/25$	0	$-2/25$	-1	1	**8**
z_j		50	40	$\frac{70 + 3M}{25}$	0	$\frac{130 + 2M}{25}$	M	$-M$	**1980 − 8*M***
$c_j - z_j$		0	0	$\frac{-70 - 3M}{25}$	0	$\frac{-130 - 2M}{25}$	$-M$	0	

If an artificial variable is positive, the solution is not feasible for the real problem.

Note that $c_j - z_j \le 0$ for all the variables; therefore, according to the optimality criterion, it should be the optimal solution. But this solution is *not feasible* for the modified HighTech problem because the artificial variable $a_4 = 8$ appears in the solution. The solution $x_1 = 30$ and $x_2 = 12$ results in a combined total production of 42 units instead of the constraint 4 requirement of at least 50 units. The fact that the artificial variable is in solution at a value of $a_4 = 8$ tells us that the final solution violates the fourth constraint ($1x_1 + 1x_2 \ge 50$) by eight units.

If management is interested in knowing which of the first three constraints is preventing us from satisfying the total production requirement, a partial answer can be obtained from the final simplex tableau. Note that $s_2 = 8$, but that s_1 and s_3 are zero. This tells us that the assembly time and warehouse capacity constraints are binding. Because not enough assembly time and warehouse space are available, we cannot satisfy the minimum combined total production requirement.

The management implications here are that additional assembly time and/or warehouse space must be made available to satisfy the total production requirement. If more time and/or space cannot be made available, management will have to relax the total production requirement by at least eight units.

Try Problem 23 to practice recognizing when there is no feasible solution to a problem using the simplex method.

In summary, a linear program is infeasible if no solution satisfies all the constraints simultaneously. *We recognize infeasibility when one or more of the artificial variables remain in the final solution at a positive value.* In closing, we note that linear programming problems with all $\leq$ constraints and nonnegative right-hand sides will always have a feasible solution. Because it is not necessary to introduce artificial variables to set up the initial simplex tableau for these types of problems, the final solution cannot possibly contain an artificial variable.

Unboundedness

Usually a constraint has been overlooked if unboundedness occurs.

For maximization problems, we say that a linear program is unbounded if the value of the solution may be made infinitely large without violating any constraints. Thus, when unboundedness occurs, we can generally look for an error in the formulation of the problem.

The coefficients in the column of the $\bar{A}$ matrix associated with the incoming variable indicate how much each of the current basic variables will decrease if one unit of the incoming variable is brought into solution. Suppose then, that for a particular linear programming problem, we reach a point where the rule for determining which variable should enter the basis results in the decision to enter variable x_2. Assume that for this variable, $c_2 - z_2 = 5$, and that all $\bar{a}_{i2}$ in column 2 are ≤ 0. Thus, each unit of x_2 brought into solution increases the objective function by five units. Furthermore, because $\bar{a}_{i2} \leq 0$ for all i, none of the current basic variables will be driven to zero, no matter how many units of x_2 we introduce. Thus, we can introduce an infinite amount of x_2 into solution and still maintain feasibility. Because each unit of x_2 increases the objective function by 5, we will have an unbounded solution. Hence, *the way we recognize the unbounded situation is that all the $\bar{a}_{ij}$ are less than or equal to zero in the column associated with the incoming variable.*

To illustrate this concept, let us consider the following example of an unbounded problem.

$$\begin{array}{lrcl}
\text{Max} & 20x_1 + 10x_2 & & \\
\text{s.t.} & 1x_1 & \geq & 2 \\
 & 1x_2 & \leq & 5 \\
 & x_1, x_2 & \geq & 0
\end{array}$$

We subtract a surplus variable s_1 from the first constraint equation and add a slack variable s_2 to the second constraint equation to obtain the standard-form representation. We then add an artificial variable a_1 to the first constraint equation to obtain the tableau form. In the initial simplex tableau the basic variables are a_1 and s_2. After bringing in x_1 and removing a_1 at the first iteration, the simplex tableau is as follows:

		x_1	x_2	s_1	s_2	
Basis	c_B	20	10	0	0	
x_1	20	1	0	−1	0	**2**
s_2	0	0	1	0	1	**5**
z_j		20	0	−20	0	**40**
$c_j - z_j$		0	10	20	0	

Because s_1 has the largest positive $c_j - z_j$, we know we can increase the value of the objective function most rapidly by bringing s_1 into the basis. But $\bar{a}_{13} = -1$ and $\bar{a}_{23} = 0$; hence, we cannot form the ratio $\bar{b}_i/\bar{a}_{i3}$ for any $\bar{a}_{i3} > 0$ because no values of $\bar{a}_{i3}$ are greater than zero.

This result indicates that the solution to the linear program is unbounded because each unit of s_1 that is brought into solution provides one extra unit of x_1 (since $\bar{a}_{13} = -1$) and drives zero units of s_2 out of solution (since $\bar{a}_{23} = 0$). Because s_1 is a surplus variable and can be interpreted as the amount of x_1 over the minimum amount required, the simplex tableau indicates we can introduce as much of s_1 as we desire without violating any constraints; the interpretation is that we can make as much as we want above the minimum amount of x_1 required. Because the objective function coefficient associated with x_1 is positive, there will be no upper bound on the value of the objective function.

In summary, a maximization linear program is unbounded if it is possible to make the value of the optimal solution as large as desired without violating any of the constraints. When employing the simplex method, an unbounded linear program exists if *at some iteration, the simplex method tells us to introduce variable j into the solution and all the* $\bar{a}_{ij}$ *are less than or equal to zero in the jth column.*

Try Problem 25 for another example of an unbounded problem.

We emphasize that the case of an unbounded solution will never occur in real cost minimization or profit maximization problems because it is not possible to reduce costs to minus infinity or to increase profits to plus infinity. Thus, if we encounter an unbounded solution to a linear programming problem, we should carefully reexamine the formulation of the problem to determine whether a formulation error has occurred.

Alternative Optimal Solutions

A linear program with two or more optimal solutions is said to have alternative optimal solutions. When using the simplex method, we cannot recognize that a linear program has alternative optimal solutions until the final simplex tableau is reached. Then if the linear program has alternative optimal solutions, $c_j - z_j$ will equal zero for one or more nonbasic variables.

To illustrate the case of alternative optimal solutions when using the simplex method, consider changing the objective function for the HighTech problem from $50x_1 + 40x_2$ to $30x_1 + 50x_2$; in doing so, we obtain the revised linear program:

$$\begin{aligned}
\text{Max} \quad & 30x_1 + 50x_2 \\
\text{s.t.} \quad & \\
& 3x_1 + 5x_2 \le 150 \\
& \qquad\;\; 1x_2 \le 20 \\
& 8x_1 + 5x_2 \le 300 \\
& x_1, x_2 \ge 0
\end{aligned}$$

The final simplex tableau for this problem is shown here:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	30	50	0	0	0	
x_2	50	0	1	0	1	0	**20**
s_3	0	0	0	$-8/3$	$25/3$	1	**$200/3$**
x_1	30	1	0	$1/3$	$-5/3$	0	**$50/3$**
z_j		30	50	10	0	0	**1500**
$c_j - z_j$		0	0	-10	0	0	

All values in the net evaluation row are less than or equal to zero, indicating that an optimal solution has been found. This solution is given by $x_1 = 50/3$, $x_2 = 20$, $s_1 = 0$, $s_2 = 0$, and $s_3 = 200/3$. The value of the objective function is 1500.

In looking at the net evaluation row in the optimal simplex tableau, we see that the $c_j - z_j$ value for nonbasic variable s_2 is equal to zero. It indicates that the linear program may have alternative optimal solutions. In other words, because the net evaluation row entry for s_2 is zero, we can introduce s_2 into the basis without changing the value of the solution. The tableau obtained after introducing s_2 follows:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	30	50	0	0	0	
x_2	50	0	1	$8/25$	0	$-3/25$	**12**
s_2	0	0	0	$-8/25$	1	$3/25$	**8**
x_1	30	1	0	$-5/25$	0	$5/25$	**30**
z_j		30	50	10	0	0	**1500**
$c_j - z_j$		0	0	-10	0	0	

Try Problem 24 for another example of alternative optimal solutions.

As shown, we have a different basic feasible solution: $x_1 = 30$, $x_2 = 12$, $s_1 = 0$, $s_2 = 8$, and $s_3 = 0$. However, this new solution is also optimal because $c_j - z_j \leq 0$ for all j. Another way to confirm that this solution is still optimal is to note that the value of the solution has remained equal to 1500.

In summary, *when using the simplex method, we can recognize the possibility of alternative optimal solutions if $c_j - z_j$ equals zero for one or more of the nonbasic variables in the final simplex tableau.*

Degeneracy

A linear program is said to be degenerate if one or more of the basic variables have a value of zero. **Degeneracy** does not cause any particular difficulties for the graphical solution procedure; however, degeneracy can theoretically cause difficulties when the simplex method is used to solve a linear programming problem.

To see how a degenerate linear program could occur, consider a change in the right-hand-side value of the assembly time constraint for the HighTech problem. For example, what if the number of hours available had been 175 instead of 150? The modified linear program is shown here.

$$
\begin{array}{lrll}
\text{Max} & 50x_1 + 40x_2 & & \\
\text{s.t.} & & & \\
& 3x_1 + \quad 5x_2 \leq 175 & & \text{Assembly time increased to 175 hours} \\
& 1x_2 \leq \quad 20 & & \text{Portable display} \\
& 8x_1 + \quad 5x_2 \leq 300 & & \text{Warehouse space} \\
& x_1, x_2 \geq 0 & &
\end{array}
$$

The simplex tableau after one iteration is as follows:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
s_1	0	0	$25/8$	1	0	$-3/8$	**$125/2$**
s_2	0	0	1	0	1	0	**20**
x_1	50	1	$5/8$	0	0	$1/8$	**$75/2$**
z_j		50	$250/8$	0	0	$50/8$	**1875**
$c_j - z_j$		0	$70/8$	0	0	$-50/8$	

The entries in the net evaluation row indicate that x_2 should enter the basis. By calculating the appropriate ratios to determine the pivot row, we obtain

$$\frac{\bar{b}_1}{\bar{a}_{12}} = \frac{125/2}{25/8} = 20$$

$$\frac{\bar{b}_2}{\bar{a}_{22}} = \frac{20}{1} = 20$$

$$\frac{\bar{b}_3}{\bar{a}_{32}} = \frac{75/2}{5/8} = 60$$

We see that the first and second rows tie, which indicates that we will have a degenerate basic feasible solution at the next iteration. Recall that in the case of a tie, we follow the convention of selecting the uppermost row as the pivot row. Here, it means that s_1 will leave the basis. But from the tie for the minimum ratio we see that the basic variable in row 2, s_2, will also be driven to zero. Because it does not leave the basis, we will have a basic variable with a value of zero after performing this iteration. The simplex tableau after this iteration is as follows:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	50	40	0	0	0	
x_2	40	0	1	$8/25$	0	$-3/25$	**20**
s_2	0	0	0	$-8/25$	1	$3/25$	**0**
x_1	50	1	0	$-5/25$	0	$5/25$	**25**
z_j		50	40	$70/25$	0	$130/25$	**2050**
$c_j - z_j$		0	0	$-70/25$	0	$-130/25$	

As expected, we have a basic feasible solution with one of the basic variables, s_2, equal to zero. Whenever we have a tie in the minimum $\bar{b}_i/\bar{a}_{ij}$ ratio, the next tableau will always have a basic variable equal to zero. Because we are at the optimal solution in the preceding case, we do not care that s_2 is in solution at a zero value. However, if degeneracy occurs at some iteration prior to reaching the optimal solution, it is theoretically possible for the simplex method to cycle; that is, the procedure could possibly alternate between the same set of nonoptimal basic feasible solutions and never reach the optimal solution. Cycling has not proven to be a significant difficulty in practice. Therefore, we do not recommend introducing any special steps into the simplex method to eliminate the possibility that degeneracy will occur. If while performing the iterations of the simplex algorithm a tie occurs for the minimum $\bar{b}_i/\bar{a}_{ij}$ ratio, then we recommend simply selecting the upper row as the pivot row.

NOTES AND COMMENTS

1. We stated that infeasibility is recognized when the stopping rule is encountered but one or more artificial variables are in solution at a positive value. This requirement does not necessarily mean that all artificial variables must be nonbasic to have a feasible solution. An artificial variable could be in solution at a zero value.
2. An unbounded feasible region must exist for a problem to be unbounded, but it does not guarantee that a problem will be unbounded. A minimization problem may be bounded, whereas a maximization problem with the same feasible region is unbounded.

SUMMARY

In this chapter the simplex method was introduced as an algebraic procedure for solving linear programming problems. Although the simplex method can be used to solve small linear programs by hand calculations, it becomes too cumbersome as problems get larger. As a result, a computer software package must be used to solve large linear programs in any reasonable length of time. The computational procedures of most computer software packages are based on the simplex method.

We described how developing the tableau form of a linear program is a necessary step in preparing a linear programming problem for solution using the simplex method, including how to convert greater-than-or-equal-to constraints, equality constraints, and constraints with negative right-hand-side values into tableau form.

For linear programs with greater-than-or-equal-to constraints and/or equality constraints, artificial variables are used to obtain tableau form. An objective function coefficient of $-M$, where M is a very large number, is assigned to each artificial variable. If there is a feasible solution to the real problem, all artificial variables will be driven out of solution (or to zero) before the simplex method reaches its optimality criterion. The iterations required to remove the artificial variables from solution constitute what is called phase I of the simplex method.

Two techniques were mentioned for solving minimization problems. The first approach involved changing the rule for introducing a variable into solution and changing the optimality criterion. The second approach involved multiplying the objective function by -1 to obtain an equivalent maximization problem. With this change, any minimization problem can be solved using the steps required for a maximization problem, but the value of the optimal solution must be multiplied by -1 to obtain the optimal value of the original minimization problem.

As a review of the material in this chapter we now present a detailed step-by-step procedure for solving linear programs using the simplex method.

Step 1. Formulate a linear programming model of the problem.
Step 2. Define an equivalent linear program by performing the following operations:
  a. Multiply each constraint with a negative right-hand-side value by -1, and change the direction of the constraint inequality.
  b. For a minimization problem, convert the problem to an equivalent maximization problem by multiplying the objective function by -1.

Step 3. Set up the standard form of the linear program by adding appropriate slack and surplus variables.

Step 4. Set up the tableau form of the linear program to obtain an initial basic feasible solution. All linear programs must be set up this way before the initial simplex tableau can be obtained.

Step 5. Set up the initial simplex tableau to keep track of the calculations required by the simplex method.

Step 6. Choose the nonbasic variable with the largest $c_j - z_j$ to bring into the basis. The column associated with that variable is the pivot column.

Step 7. Choose as the pivot row that row with the smallest ratio of $\bar{b}_i/\bar{a}_{ij}$ for $\bar{a}_{ij} > 0$. This ratio is used to determine which variable will leave the basis when variable j enters the basis. This ratio also indicates how many units of variable j can be introduced into solution before the basic variable in the ith row equals zero.

Step 8. Perform the necessary elementary row operations to convert the pivot column to a unit column.

  a. Divide each element in the pivot row by the pivot element. The result is a new pivot row containing a 1 in the pivot column.

  b. Obtain zeroes in all other positions of the pivot column by adding or subtracting an appropriate multiple of the new pivot row.

Step 9. Test for optimality. If $c_j - z_j \leq 0$ for all columns, we have the optimal solution. If not, return to step 6.

In Section 17.8 we discussed how the special cases of infeasibility, unboundedness, alternative optimal solutions, and degeneracy can occur when solving linear programming problems with the simplex method.

GLOSSARY

Simplex method An algebraic procedure for solving linear programming problems. The simplex method uses elementary row operations to iterate from one basic feasible solution (extreme point) to another until the optimal solution is reached.

Basic solution Given a linear program in standard form, with n variables and m constraints, a basic solution is obtained by setting $n - m$ of the variables equal to zero and solving the constraint equations for the values of the other m variables. If a unique solution exists, it is a basic solution.

Nonbasic variable One of $n - m$ variables set equal to zero in a basic solution.

Basic variable One of the m variables not required to equal zero in a basic solution.

Basic feasible solution A basic solution that is also feasible; that is, it satisfies the nonnegativity constraints. A basic feasible solution corresponds to an extreme point.

Tableau form The form in which a linear program must be written before setting up the initial simplex tableau. When a linear program is written in tableau form, its A matrix contains m unit columns corresponding to the basic variables, and the values of these basic variables are given by the values in the b column. A further requirement is that the entries in the b column be greater than or equal to zero.

Simplex tableau A table used to keep track of the calculations required by the simplex method.

Unit column or unit vector A vector or column of a matrix that has a zero in every position except one. In the nonzero position there is a 1. There is a unit column in the simplex tableau for each basic variable.

Basis The set of variables that are not restricted to equal zero in the current basic solution. The variables that make up the basis are termed basic variables, and the remaining variables are called nonbasic variables.

Net evaluation row The row in the simplex tableau that contains the value of $c_j - z_j$ for every variable (column).

Iteration The process of moving from one basic feasible solution to another.

Pivot element The element of the simplex tableau that is in both the pivot row and the pivot column.

Pivot column The column in the simplex tableau corresponding to the nonbasic variable that is about to be introduced into solution.

Pivot row The row in the simplex tableau corresponding to the basic variable that will leave the solution.

Elementary row operations Operations that may be performed on a system of simultaneous equations without changing the solution to the system of equations.

Artificial variable A variable that has no physical meaning in terms of the original linear programming problem, but serves merely to enable a basic feasible solution to be created for starting the simplex method. Artificial variables are assigned an objective function coefficient of $-M$, where M is a very large number.

Phase I When artificial variables are present in the initial simplex tableau, phase I refers to the iterations of the simplex method that are required to eliminate the artificial variables. At the end of phase I, the basic feasible solution in the simplex tableau is also feasible for the real problem.

Degeneracy When one or more of the basic variables has a value of zero.

PROBLEMS

SELF test

1. Consider the following system of linear equations:

$$\begin{aligned} 3x_1 + x_2 \qquad &= 6 \\ 2x_1 + 4x_2 + x_3 &= 12 \end{aligned}$$

a. Find the basic solution with $x_1 = 0$.
b. Find the basic solution with $x_2 = 0$.
c. Find the basic solution with $x_3 = 0$.
d. Which of the preceding solutions would be basic feasible solutions for a linear program?

2. Consider the following linear program:

$$\begin{aligned} \text{Max} \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & \\ & x_1 + 5x_2 \leq 10 \\ & 2x_1 + 6x_2 \leq 16 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- **a.** Write the problem in standard form.
- **b.** How many variables will be set equal to zero in a basic solution for this problem?
- **c.** Find all the basic solutions, and indicate which are also feasible.
- **d.** Find the optimal solution by computing the value of each basic feasible solution.

3. Consider the following linear program:

$$\begin{array}{ll} \text{Max} & 5x_1 + 9x_2 \\ \text{s.t.} & \\ & \tfrac{1}{2}x_1 + 1x_2 \leq 8 \\ & 1x_1 + 1x_2 \geq 10 \\ & \tfrac{1}{4}x_1 + \tfrac{3}{2}x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

- **a.** Write the problem in standard form.
- **b.** How many variables will be set equal to zero in a basic solution for this problem? Explain.
- **c.** Find the basic solution that corresponds to s_1 and s_2 equal to zero.
- **d.** Find the basic solution that corresponds to x_1 and s_3 equal to zero.
- **e.** Are your solutions for parts (c) and (d) basic feasible solutions? Extreme-point solutions? Explain.
- **f.** Use the graphical approach to identify the solutions found in parts (c) and (d). Do the graphical results agree with your answer to part (e)? Explain.

SELF test

4. Consider the following linear programming problem:

$$\begin{array}{ll} \text{Max} & 60x_1 + 90x_2 \\ \text{s.t.} & \\ & 15x_1 + 45x_2 \leq 90 \\ & 5x_1 + 5x_2 \leq 20 \\ & x_1, x_2 \geq 0 \end{array}$$

- **a.** Write the problem in standard form.
- **b.** Develop the portion of the simplex tableau involving the objective function coefficients, the coefficients of the variables in the constraints, and the constants for the right-hand sides.

SELF test

5. A partially completed initial simplex tableau is given:

		x_1	x_2	s_1	s_2	
Basis	c_B	5	9	0	0	
s_1	0	10	9	1	0	**90**
s_2	0	−5	3	0	1	**15**
z_j						
$c_j - z_j$						

- **a.** Complete the initial tableau.
- **b.** Which variable would be brought into solution at the first iteration?
- **c.** Write the original linear program.

SELF test

6. The following partial initial simplex tableau is given:

		x_1	x_2	x_3	s_1	s_2	s_3	
Basis	c_B	5	20	25	0	0	0	
		2	1	0	1	0	0	**40**
		0	2	1	0	1	0	**30**
		3	0	$-\frac{1}{2}$	0	0	1	**15**
z_j								
$c_j - z_j$								

a. Complete the initial tableau.
b. Write the problem in tableau form.
c. What is the initial basis? Does this basis correspond to the origin? Explain.
d. What is the value of the objective function at this initial solution?
e. For the next iteration, which variable should enter the basis, and which variable should leave the basis?
f. How many units of the entering variable will be in the next solution? Before making this first iteration, what do you think will be the value of the objective function after the first iteration?
g. Find the optimal solution using the simplex method.

7. Solve the following linear program using the graphical approach:

$$
\begin{aligned}
\text{Max}\quad & 4x_1 + 5x_2 \\
\text{s.t.}\quad & \\
& 2x_1 + 2x_2 \le 20 \\
& 3x_1 + 7x_2 \le 42 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Put the linear program in tableau form, and solve using the simplex method. Show the sequence of extreme points generated by the simplex method on your graph.

8. Recall the problem for Par, Inc., introduced in Section 2.1. The mathematical model for this problem is restated as follows:

$$
\begin{aligned}
\text{Max}\quad & 10x_1 + 9x_2 \\
\text{s.t.}\quad & \\
& \tfrac{7}{10}x_1 + 1x_2 \le 630 \quad \text{Cutting and dyeing} \\
& \tfrac{1}{2}x_1 + \tfrac{5}{6}x_2 \le 600 \quad \text{Sewing} \\
& 1x_1 + \tfrac{2}{3}x_2 \le 708 \quad \text{Finishing} \\
& \tfrac{1}{10}x_1 + \tfrac{1}{4}x_2 \le 135 \quad \text{Inspection and packaging} \\
& x_1, x_2 \ge 0
\end{aligned}
$$

where

x_1 = number of standard bags produced
x_2 = number of deluxe bags produced

a. Use the simplex method to determine how many bags of each model Par should manufacture.
b. What is the profit Par can earn with these production quantities?

c. How many hours of production time will be scheduled for each operation?
d. What is the slack time in each operation?

9. RMC, Inc., is a small firm that produces a variety of chemical products. In a particular production process, three raw materials are blended (mixed together) to produce two products: a fuel additive and a solvent base. Each ton of fuel additive is a mixture of $\frac{2}{5}$ ton of material 1 and $\frac{3}{5}$ ton of material 3. A ton of solvent base is a mixture of $\frac{1}{2}$ ton of material 1, $\frac{1}{5}$ ton of material 2, and $\frac{3}{10}$ ton of material 3. After deducting relevant costs, the profit contribution is $40 for every ton of fuel additive produced and $30 for every ton of solvent base produced.

RMC's production is constrained by a limited availability of the three raw materials. For the current production period, RMC has available the following quantities of each raw material:

Raw Material	**Amount Available for Production**
Material 1	20 tons
Material 2	5 tons
Material 3	21 tons

Assuming that RMC is interested in maximizing the total profit contribution, the problem formulation is shown here:

$$\begin{array}{ll} \text{Max} & 40x_1 + 30x_2 \\ \text{s.t.} & \\ & \frac{2}{5}x_1 + \frac{1}{2}x_2 \le 20 \quad \text{Material 1} \\ & \qquad\quad \frac{1}{5}x_2 \le 5 \quad \text{Material 2} \\ & \frac{3}{5}x_1 + \frac{3}{10}x_2 \le 21 \quad \text{Material 3} \\ & x_1, x_2 \ge 0 \end{array}$$

where

x_1 = tons of fuel additive produced
x_2 = tons of solvent base produced

Solve the RMC problem using the simplex method. At each iteration, locate the basic feasible solution found by the simplex method on the graph of the feasible region.

10. Solve the following linear program:

$$\begin{array}{ll} \text{Max} & 5x_1 + 5x_2 + 24x_3 \\ \text{s.t.} & \\ & 15x_1 + 4x_2 + 12x_3 \le 2800 \\ & 15x_1 + 8x_2 \qquad\quad \le 6000 \\ & x_1 \qquad\quad + 8x_3 \le 1200 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

11. Solve the following linear program using both the graphical and the simplex methods:

$$\begin{aligned} \text{Max} \quad & 2x_1 + 8x_2 \\ \text{s.t.} \quad & \\ & 3x_1 + 9x_2 \leq 45 \\ & 2x_1 + 1x_2 \geq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Show graphically how the simplex method moves from one basic feasible solution to another. Find the coordinates of all extreme points of the feasible region.

12. Suppose a company manufactures three products from two raw materials. The amount of raw material in each unit of each product is given.

Raw Material	Product A	Product B	Product C
I	7 lb	6 lb	3 lb
II	5 lb	4 lb	2 lb

If the company has available 100 pounds of material I and 200 pounds of material II, and if the profits for the three products are $20, $20, and $15, respectively, how much of each product should be produced to maximize profits?

13. Liva's Lumber, Inc., manufactures three types of plywood. The following table summarizes the production hours per unit in each of three production operations and other data for the problem.

	Operations (hours)			
Plywood	I	II	III	Profit/Unit
Grade A	2	2	4	$40
Grade B	5	5	2	$30
Grade X	10	3	2	$20
Maximum time available	900	400	600	

How many units of each grade of lumber should be produced?

14. Ye Olde Cording Winery in Peoria, Illinois, makes three kinds of authentic German wine: Heidelberg Sweet, Heidelberg Regular, and Deutschland Extra Dry. The raw materials, labor, and profit for a gallon of each of these wines are summarized here:

Wine	Grade A Grapes (bushels)	Grade B Grapes (bushels)	Sugar (pounds)	Labor (hours)	Profit/ Gallon
Heidelberg Sweet	1	1	2	2	$1.00
Heidelberg Regular	2	0	1	3	$1.20
Deutschland Extra Dry	0	2	0	1	$2.00

If the winery has 150 bushels of grade A grapes, 150 bushels of grade B grapes, 80 pounds of sugar, and 225 labor-hours available during the next week, what product mix of wines will maximize the company's profit?

a. Solve using the simplex method.
b. Interpret all slack variables.
c. An increase in which resources could improve the company's profit?

SELF test

15. Set up the tableau form for the following linear program (do not attempt to solve):

$$\begin{array}{ll}\text{Max} & 4x_1 + 2x_2 - 3x_3 + 5x_4 \\ \text{s.t.} & \\ & 2x_1 - 1x_2 + 1x_3 + 2x_4 \geq 50 \\ & 3x_1 \qquad\quad - 1x_3 + 2x_4 \leq 80 \\ & 1x_1 + 1x_2 \qquad\quad + 1x_4 = 60 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

16. Set up the tableau form for the following linear program (do not attempt to solve):

$$\begin{array}{ll}\text{Min} & 4x_1 + 5x_2 + 3x_3 \\ \text{s.t.} & \\ & 4x_1 \qquad\quad + 2x_3 \geq 20 \\ & \qquad 1x_2 - 1x_3 \leq -8 \\ & 1x_1 - 2x_2 \qquad\quad = -5 \\ & 2x_1 + 1x_2 + 1x_3 \leq 12 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

SELF test

17. Solve the following linear program:

$$\begin{array}{ll}\text{Min} & 3x_1 + 4x_2 + 8x_3 \\ \text{s.t.} & \\ & 4x_1 + 2x_2 \qquad\quad \geq 12 \\ & \qquad 4x_2 + 8x_3 \geq 16 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

18. Solve the following linear program:

$$\begin{array}{ll}\text{Min} & 84x_1 + 4x_2 + 30x_3 \\ \text{s.t.} & \\ & 8x_1 + 1x_2 + 3x_3 \leq 240 \\ & 16x_1 + 1x_2 + 7x_3 \geq 480 \\ & 8x_1 - 1x_2 + 4x_3 \geq 160 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

19. Captain John's Yachts, Inc., located in Fort Lauderdale, Florida, rents three types of ocean-going boats: sailboats, cabin cruisers, and Captain John's favorite, the luxury yachts. Captain John advertises his boats with his famous "you rent—we pilot" slogan, which means that the company supplies the captain and crew for each rented boat. Each rented boat has one captain, of course, but the crew sizes (deck hands, galley hands, etc.) differ. The crew requirements, in addition to a captain, are one for sailboats, two for cabin cruisers, and three for yachts. Ten employees are captains, and an additional 18 employees fill the various crew positions. Currently, Captain John has rental requests for all of his boats: four sailboats, eight cabin cruisers, and three luxury yachts. If Captain John's daily profit contribution is $50 for sailboats, $70 for cruisers, and $100 for luxury yachts, how many boats of each type should he rent?

20. The Our-Bags-Don't-Break (OBDB) plastic bag company manufactures three plastic refuse bags for home use: a 20-gallon garbage bag, a 30-gallon garbage bag, and a 33-gallon leaf-and-grass bag. Using purchased plastic material, three operations are required to produce each end product: cutting, sealing, and packaging. The production time required to process each type of bag in every operation and the maximum production time available for each operation are shown (note that the production time figures in this table are per box of each type of bag).

	Production Time (seconds/box)		
Type of Bag	**Cutting**	**Sealing**	**Packaging**
20 gallons	2	2	3
30 gallons	3	2	4
33 gallons	3	3	5
Time available	2 hours	3 hours	4 hours

If OBDB's profit contribution is $0.10 for each box of 20-gallon bags produced, $0.15 for each box of 30-gallon bags, and $0.20 for each box of 33-gallon bags, what is the optimal product mix?

21. Kirkman Brothers ice cream parlors sell three different flavors of Dairy Sweet ice milk: chocolate, vanilla, and banana. Due to extremely hot weather and a high demand for its products, Kirkman has run short of its supply of ingredients: milk, sugar, and cream. Hence, Kirkman will not be able to fill all the orders received from its retail outlets, the ice cream parlors. Due to these circumstances, Kirkman decided to make the most profitable amounts of the three flavors, given the constraints on supply of the basic ingredients. The company will then ration the ice milk to the retail outlets.

Kirkman collected the following data on profitability of the various flavors, availability of supplies, and amounts required for each flavor.

		Usage/Gallon		
Flavor	**Profit/ Gallon**	**Milk (gallons)**	**Sugar (pounds)**	**Cream (gallons)**
Chocolate	$1.00	0.45	0.50	0.10
Vanilla	$0.90	0.50	0.40	0.15
Banana	$0.95	0.40	0.40	0.20
Maximum available		200	150	60

Determine the optimal product mix for Kirkman Brothers. What additional resources could be used profitably?

22. Uforia Corporation sells two brands of perfume: Incentive and Temptation No. 1. Uforia sells exclusively through department stores and employs a three-person sales staff to call on its customers. The amount of time necessary for each sales representative to sell one case of each product varies with experience and ability. Data on the average time for each of Uforia's three sales representatives is presented here.

	Average Sales Time per Case (minutes)	
Salesperson	**Incentive**	**Temptation No. 1**
John	10	15
Brenda	15	10
Red	12	6

Each sales representative spends approximately 80 hours per month in the actual selling of these two products. Cases of Incentive and Temptation No. 1 sell at profits of $30 and $25, respectively. How many cases of each perfume should each person sell during the next month to maximize the firm's profits? (*Hint:* Let x_1 = number of cases of Incentive sold by John, x_2 = number of cases of Temptation No. 1 sold by John, x_3 = number of cases of Incentive sold by Brenda, and so on.)

Note: In Problems 23–29, we provide examples of linear programs that result in one or more of the following situations:

- Optimal solution
- Infeasible solution
- Unbounded solution
- Alternative optimal solutions
- Degenerate solution

For each linear program, determine the solution situation that exists, and indicate how you identified each situation using the simplex method. For the problems with alternative optimal solutions, calculate at least two optimal solutions.

SELF test

23.

$$\begin{array}{ll} \text{Max} & 4x_1 + 8x_2 \\ \text{s.t.} & \\ & 2x_1 + 2x_2 \leq 10 \\ & -1x_1 + 1x_2 \geq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

SELF test

24.

$$\begin{array}{ll} \text{Min} & 3x_1 + 3x_2 \\ \text{s.t.} & \\ & 2x_1 + 0.5x_2 \geq 10 \\ & 2x_1 \geq 4 \\ & 4x_1 + 4x_2 \geq 32 \\ & x_1, x_2 \geq 0 \end{array}$$

SELF test

25.

$$\begin{array}{ll} \text{Min} & 1x_1 + 1x_2 \\ \text{s.t.} & \\ & 8x_1 + 6x_2 \geq 24 \\ & 4x_1 + 6x_2 \geq -12 \\ & 2x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

26.

$$\begin{array}{ll} \text{Max} & 2x_1 + 1x_2 + 1x_3 \\ \text{s.t.} & \\ & 4x_1 + 2x_2 + 2x_3 \geq 4 \\ & 2x_1 + 4x_2 \leq 20 \\ & 4x_1 + 8x_2 + 2x_3 \leq 16 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

27.

$$\begin{aligned} \text{Max} \quad & 2x_1 + 4x_2 \\ \text{s.t.} \quad & \\ & 1x_1 + \tfrac{1}{2}x_2 \le 10 \\ & 1x_1 + 1x_2 = 12 \\ & 1x_1 + \tfrac{3}{2}x_2 \le 18 \\ & x_1, x_2 \ge 0 \end{aligned}$$

28.

$$\begin{aligned} \text{Min} \quad & -4x_1 + 5x_2 + 5x_3 \\ \text{s.t.} \quad & \\ & 1x_2 + 1x_3 \ge 2 \\ & -1x_1 + 1x_2 + 1x_3 \ge 1 \\ & -1x_3 \ge 1 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

29. Solve the following linear program and identify any alternative optimal solutions.

$$\begin{aligned} \text{Max} \quad & 120x_1 + 80x_2 + 14x_3 \\ \text{s.t.} \quad & \\ & 4x_1 + 8x_2 + x_3 \le 200 \\ & 2x_2 + 1x_3 \le 300 \\ & 32x_1 + 4x_2 + 2x_3 = 400 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

30. Supersport Footballs, Inc., manufactures three kinds of footballs: an All-Pro model, a College model, and a High School model. All three footballs require operations in the following departments: cutting and dyeing, sewing, and inspection and packaging. The production times and maximum production availabilities are shown here.

	Production Time (minutes)		
Model	**Cutting and Dyeing**	**Sewing**	**Inspection and Packaging**
All-Pro	12	15	3
College	10	15	4
High School	8	12	2
Time available	300 hours	200 hours	100 hours

Current orders indicate that at least 1000 All-Pro footballs must be manufactured.

a. If Supersport realizes a profit contribution of \$3 for each All-Pro model, \$5 for each College model, and \$4 for each High School model, how many footballs of each type should be produced? What occurs in the solution of this problem? Why?

b. If Supersport can increase sewing time to 300 hours and inspection and packaging time to 150 hours by using overtime, what is your recommendation?

Self-Test Solutions and Answers to Even-Numbered Problems

Chapter 17

1. a. With $x_1 = 0$, we have

$$\begin{aligned} x_2 \quad &= 6 \qquad (1) \\ 4x_2 + x_3 &= 12 \qquad (2) \end{aligned}$$

From (1), we have $x_2 = 6$; substituting for x_2 in (2) yields

$$\begin{aligned} 4(6) + x_3 &= 12 \\ x_3 &= 12 - 24 = -12 \end{aligned}$$

Basic solution: $x_1 = 0, x_2 = 6, x_3 = -12$

b. With $x_2 = 0$, we have

$$\begin{aligned} 3x_1 \quad &= 6 \qquad (3) \\ 2x_1 + x_3 &= 12 \qquad (4) \end{aligned}$$

From (3), we find $x_1 = 2$; substituting for x_1 in (4) yields

$$\begin{aligned} 2(2) + x_3 &= 12 \\ x_3 &= 12 - 4 = 8 \end{aligned}$$

Basic solution: $x_1 = 2, x_2 = 0, x_3 = 8$

c. With $x_3 = 0$, we have

$$\begin{aligned} 3x_1 + x_2 &= 6 \qquad (5) \\ 2x_1 + 4x_2 &= 12 \qquad (6) \end{aligned}$$

Multiplying (6) by $\frac{3}{2}$ and subtracting from (5) yields

$$\begin{aligned} 3x_1 + x_2 &= 6 \\ -(3x_1 + 6x_2) &= -18 \\ \hline -5x_2 &= -12 \\ x_2 &= \tfrac{12}{5} \end{aligned}$$

Substituting $x_2 = \frac{12}{5}$ into (5) yields

$$\begin{aligned} 3x_1 + \tfrac{12}{5} &= 6 \\ 3x_1 &= \tfrac{18}{5} \\ x_1 &= \tfrac{6}{5} \end{aligned}$$

Basic solution: $x_1 = \frac{6}{5}, x_2 = \frac{12}{5}, x_3 = 0$

d. The basic solutions found in parts (b) and (c) are basic feasible solutions. The one in part (a) is not because $x_3 = -12$.

2. a. Max $\quad x_1 + 2x_2$

s.t.

$$\begin{aligned} x_1 + 5x_2 + s_1 \qquad &= 10 \\ 2x_1 + 6x_2 \qquad + s_2 &= 16 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

b. 2

c. $x_1 = 0, x_2 = 0, s_1 = 10, s_2 = 16$; feasible
$x_1 = 0, x_2 = 2, s_1 = 0, s_2 = 4$; feasible
$x_1 = 0, x_2 = \frac{8}{3}, s_1 = -\frac{10}{3}, s_2 = 0$; not feasible
$x_1 = 10, x_2 = 0, s_1 = 0, s_2 = -4$; not feasible
$x_1 = 8, x_2 = 0, s_1 = 2, s_2 = 0$; feasible
$x_1 = 5, x_2 = 1, s_1 = 0, s_2 = 0$; feasible

d. $x_1 = 8, x_2 = 0$; Value $= 8$

4. a. Standard form:

Max $\quad 60x_1 + 90x_2$

s.t.

$$\begin{aligned} 15x_1 + 45x_2 + s_1 \qquad &= 90 \\ 5x_1 + 5x_2 \qquad + s_2 &= 20 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

b. Partial initial simple tableau:

x_1	x_2	s_1	s_2	
60	90	0	0	
15	45	1	0	**90**
5	5	0	1	**20**

5. a. Initial tableau:

		x_1	x_2	s_1	s_2	
Basis	c_B	5	9	0	0	
s_1	0	10	9	1	0	**90**
s_2	0	−5	3	0	1	**15**
z_j		0	0	0	0	**0**
$c_j - z_j$		5	9	0	0	

b. Introduce x_2 at the first iteration

c. Max $\quad 5x_1 + 9x_2$

s.t.

$$\begin{aligned} 10x_1 + 9x_2 &\leq 90 \\ -5x_1 + 3x_2 &\leq 15 \\ x_1, x_2 &\geq 0 \end{aligned}$$

6. a.

z_j	0	0	0	0	0	0	**0**
$c_j - z_j$	5	20	25	0	0	0	

b. Max $\quad 5x_1 + 20x_2 + 25x_3 + 0s_1 + 0s_2 + 0s_3$

s.t.

$$\begin{aligned} 2x_1 + 1x_2 \qquad\qquad + 1s_1 \qquad\qquad\qquad &= 40 \\ 2x_2 + \phantom{\tfrac{1}{2}}1x_3 \qquad\qquad + 1s_2 \qquad &= 30 \\ 3x_1 \qquad\qquad - \tfrac{1}{2}x_3 \qquad\qquad\qquad + 1s_3 &= 15 \\ x_1, x_2, x_3, s_1, s_2, s_3 &\geq 0 \end{aligned}$$

c. s_1, s_2, s_3; it is the origin

- **d.** 0
- **e.** x_3 enters, s_2 leaves
- **f.** 30, 750
- **g.** $x_1 = 10, s_1 = 20$
  $x_2 = 0, s_2 = 0$, Value $= 800$
  $x_3 = 30, s_3 = 0$

8. **a.** $x_1 = 540, x_2 = 252$
b. $7668
c. 630, 480, 708, 117
d. 0, 120, 0, 18

10. $x_2 = 250, x_3 = 150, s_2 = 4000$
Value $= 4850$

12. A = 0, B = 0, C = $33\frac{1}{3}$; Profit = 500

14. **a.** $x_1 = 0, x_2 = 50, x_3 = 75$; Profit = $210
c. Grade B grapes and labor

15. Max $4x_1 + 2x_2 - 3x_3 + 5x_4 + 0s_1 - Ma_1 + 0s_2 - Ma_3$
s.t.

$$\begin{aligned}
2x_1 - 1x_2 + 1x_3 + 2x_4 - 1s_1 + 1a_1 &= 50 \\
3x_1 \qquad - 1x_3 + 2x_4 \qquad + 1s_2 &= 80 \\
1x_1 + 1x_2 \qquad + 1x_4 \qquad + 1a_3 &= 60 \\
x_1, x_2, x_3, x_4, s_1, s_2, a_1, a_3 &\geq 0
\end{aligned}$$

16.
Max $-4x_1 - 5x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_4 - Ma_1 - Ma_2 - Ma_3$
s.t.

$$\begin{aligned}
4x_1 \qquad + 2x_3 - 1s_1 \qquad + 1a_1 &= 20 \\
- 1x_2 + 1x_3 \qquad - 1s_2 \qquad + 1a_2 &= 8 \\
-1x_1 + 2x_2 \qquad + 1a_3 &= 5 \\
2x_1 + 1x_2 + 1x_3 \qquad + 1s_4 &= 12 \\
x_1, x_2, x_3, s_1, s_2, s_4, a_1, a_2, a_3 &\geq 0
\end{aligned}$$

17. Converting to a max problem and solving using the simplex method, the final simplex tableau is

		x_1	x_2	x_3	s_1	s_2	
Basis	c_B	-3	-4	-8	0	0	
x_1	-3	1	0	-1	$-\frac{1}{4}$	$\frac{1}{8}$	**1**
x_2	-4	0	1	2	0	$-\frac{1}{4}$	**4**
z_j		-3	-4	-5	$\frac{3}{4}$	$\frac{5}{8}$	**−19**
$c_j - z_j$		0	0	-3	$-\frac{3}{4}$	$-\frac{5}{8}$	

18. $x_2 = 60, x_3 = 60, s_3 = 20$; Value $= 2040$

20. 2400 boxes of 33 gallon bags
Profit = $480

22. $x_1 = 480, x_4 = 480, x_6 = 800$; Value $= 46{,}400$

23. Final simplex tableau:

		x_1	x_2	s_1	s_2	a_2	
Basis	c_B	4	8	0	0	$-M$	
x_2	8	1	1	$\frac{1}{2}$	0	0	**5**
a_2	$-M$	-2	0	$-\frac{1}{2}$	-1	1	**3**
z_j		$8 + 2M$	8	$4 + M/2$	$+M$	$-M$	**$40 - 3M$**
$c_j - z_j$		$-4 - 2M$	0	$-4 - M/2$	$-M$	0	

Infeasible; optimal solution condition is reached with the artificial variable a_2 still in the solution

24. Alternative optimal solutions:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	-3	-3	0	0	0	
s_2	0	0	0	$-\frac{4}{3}$	1	$\frac{1}{6}$	**4**
x_1	-3	1	0	$-\frac{2}{3}$	0	$\frac{1}{12}$	**4**
x_2	-3	0	1	$\frac{2}{3}$	0	$-\frac{1}{3}$	**4**
z_j		-3	-3	0	0	$\frac{3}{4}$	**−24**
$c_j - z_j$		0	0	0	0	$-\frac{3}{4}$	

↑
Indicates alternative optimal solutions exist:
$x_1 = 4, x_2 = 4, z = 24$
$x_1 = 8, x_2 = 0, z = 24$

25. Unbounded solution:

		x_1	x_2	s_1	s_2	s_3	
Basis	c_B	1	1	0	0	0	
s_3	0	$\frac{8}{3}$	0	$-\frac{1}{3}$	0	1	**4**
s_2	0	4	0	-1	1	0	**36**
x_2	1	$\frac{4}{3}$	1	$-\frac{1}{6}$	0	0	**4**
z_j		$\frac{4}{3}$	1	$-\frac{1}{6}$	0	0	**4**
$c_j - z_j$		$-\frac{1}{3}$	0	$\frac{1}{6}$	0	0	

↑
Incoming
column

26. Alternative optimal solution: $x_1 = 4, x_2 = 0, x_3 = 0$
$x_1 = 0, x_2 = 0, x_3 = 8$

28. Infeasible

30. **a.** Infeasible solution; not enough sewing time
b. Alternative optimal solutions: $x_1 = 1000, x_2 = 0$, $x_3 = 250$ or $x_1 = 1000, x_2 = 200, x_3 = 0$
Profit = $4000