Homework Assignment 5

Give written (and legible) answers to the following questions. Whenever you are asked to give an algorithm for a problem, you should present the following: the algorithm, a justification of its correctness, and a derivation of its running time. Write clear and convincing pseudo-code for your algorithms. All work must be done independently.

I. (25 points) Before attempting this question, you might find it useful to read the discussion about the dual transform of line segments on page 168 of the textbook.

1. Let $S$ be a set of $n$ line segments in the plane. Suppose we want to answer the following query: Given a query line $\ell$, how many segments in $S$ does it intersect? Formulate the query in the dual plane, and explain how it is answered.

2. A line $\ell$ that intersects all the segments of $S$ is called a transversal or stabber for $S$. See Figure 1. Give an $O(n^2)$ algorithm to decide if a stabber exists for $S$.

II. (25 points) Consider a set $P$ of $n$ points in the plane. For $k \leq \lfloor n/2 \rfloor$, point $q$ (which may or may not be in $P$) is called a $k$-splitter if every line $\ell$ passing through $q$ has at least $k$ points of $P$ lying on or above it and at least $k$ points on or below it. (For example, the point $q$ shown in the figure on the next page is a 3-splitter, since every line passing through $q$ has at least 3 points of $P$ lying on either side. But it is not a 4-splitter since a horizontal line through $q$ has only 3 points below it.) Observe that any point outside the convex hull of $P$ is a 0-splitter, whereas a point within the convex hull is a $k$-splitter with $k \geq 1$.

1. Show that for all (sufficiently large) $n$, there exists a set of $n$ points that has no $\lfloor n/2 \rfloor$ splitter.

2. The level $L_m$ of a line arrangement is the set of edges of the arrangement that have at most $m$ lines above them. These edges form an $x$-monotone polygonal chain. Prove that there exists a $k$-splitter if and only if in the dual line arrangement, levels $L_{k-1}$ and $L_{n-k}$ can be separated by a line.
III. (25 points) The Delaunay triangulation is but one example of a straight-line planar graph defined on a set \( P \) of \( n \) sites in the plane. Two others include the Gabriel graph and the relative neighborhood graph (or RNG). In both cases the vertices of the graphs are the sites of \( P \). See Problems 9.13 (for part 1) and 9.14 (for part 2) of the text.

1. In the Gabriel graph, two sites \( p_i \) and \( p_j \) are joined by an edge if the disk whose diameter is \( p_ip_j \) contains no other sites.
   (a) Prove that the Gabriel graph of \( P \) is a subgraph of the Delaunay triangulation of \( P \) (that is, every edge of the Gabriel graph must also be an edge of the Delaunay triangulation).
   (b) Exercise 9.13(b). Note that the second \( p \) in the problem statement should actually be \( P \). Also, note that the statement is an 'if and only if' condition, so you must prove it in both directions.
   (c) Exercise 9.13(c). There is a short description for this algorithm.

2. In the RNG, there is an edge joining \( p_i \) and \( p_j \) if there is no point \( p_k \in P \) that is simultaneously closer to \( p_i \) and \( p_j \) than they are to each other. That is, there is no \( p_k \) such that \( \max\{\text{dist}(p_i, p_k), \text{dist}(p_j, p_k)\} < \text{dist}(p_i, p_j) \).
   (a) Exercise 9.14(a).
   (b) Prove that the RNG of \( P \) is a subgraph of the Delaunay triangulation of \( P \).

IV. (25 points) We start with two (seemingly unrelated) definitions. A triangulation is acute if the angles of all its triangles are less than 90 degrees. A Voronoi diagram is said to be medial if each edge of the diagram contains in its interior the midpoint of the two defining sites for the edge. The diagram in the figure below fails to be medial because the Voronoi edge between \( p_1 \) and \( p_2 \) fails to contain the midpoint between its two sites (marked with an x).

1. Prove that an acute triangulation of a set of points is always Delaunay, that is, it satisfies the empty circumcircle property.
2. Prove that if a Voronoi diagram is medial, then the corresponding Delaunay triangulation is acute.

It will be useful to review the facts about chords and angles covered in class.

Please upload your homework assignment on Sakai by 11:55pm on May 6, 2019.