An Improved LP-based Approximation for Steiner Tree

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Outline

- Undirected Cut Relaxation
- Bidirected Cut Relaxation
- Directed Component Cut Relaxation
  - Bridge Lemma
  - Iterative Randomized Rounding
Steiner Tree Problem

Input:
- Complete, Metric Graph $G = (V, E)$
- Weight of edges $w_e \quad \forall e \in E$
- Set of terminals $R \subseteq V$

Output:
- Minimum cost tree which spans all the terminals in $R$
Steiner Tree Problem
Use shortest path metric
History

Hardness:
- NP hard even when edge costs $\in \{1, 2\}$ [Bern & Plassmann’ 89]
- APX Hard

Approximation:
- 2-Apx (Minimum Spanning Tree)
- ...
- $1 + \frac{\ln 3}{2} + \varepsilon < 1.55$-Apx [Robins & Zelikovsky]
Different Linear Programs:
- Undirected Cut Relaxation
- Bi-directed Cut Relaxation
- Directed Component Relaxation
Undirected Constraint
Undirected Cut Relaxation

Definition:

- $\delta(S) := \text{edges crossing cut } S$

LP of UCR:

$$\begin{align*}
\min \sum_{e} w_e x_e \\
\sum_{e \in \delta(S)} x_e \geq 1 \\
x_e \geq 0 \\
\forall S = \{S \subset V | u \in S, v \notin S\} \\
\forall e \in E
\end{align*}$$
Integrality Gap for UCR

Input

OPT_{IP} = n - 1 = 3

OPT_{LP} = \frac{n}{2} = 2

Integrality Gap = 2
Bi-directed Cut Relaxation

Definition:

- \( \delta^+(S) \) := set of edges leaving \( S \)
- \( r \) := global root terminal
Bi-directed Cut Constraint
Bi-directed Cut Constraint
Bi-directed Cut Constraint

\[ S \]
Bidirected Cut Relaxation

LP for BCR:

\[
\begin{align*}
\min & \sum_{e \in E} w_e x_e \\
\sum_{e \in \delta^+(S)} x_e & \geq 1 \quad \forall S \subseteq V \setminus \{r\}, \ S \cap R \neq 0 \\
x_e & \geq 0 \quad \forall e \in E
\end{align*}
\]

\text{theorem(Edmonds)}

If \( R = V \) \( \Rightarrow \) BCR is integral
A component is a maximal subtree whose terminals coincide with its leaves.
A $k$-restricted Steiner tree contains components with no more than $k$ terminals.

4-restricted Steiner Tree
Potential Components

We want to enumerate all possible components and determine their cost.
Choose arbitrary nodes $R(C_j)$ for component $C_j$
Potential Components

Cost of $C_j = \min \text{ Steiner tree on } R(C_j)$
Potential Directed Components

Definition:
- \( \text{sink}(C_j) := \text{root terminal of } C_j \)
- \( \text{sources}(C_j) := R(C_j) \setminus \text{sink}(C_j) \)

Direct all edges of the component to the sink node.
Component $C$ crosses $S$ when at least one source of $C \in S$ and $\text{sink}(C) \notin S$
Steiner Tree
LP for $k$-DCR

![Diagram of Steiner Tree](image-url)
LP for $k$-DCR:

$$\min \sum_{j} c(C_j) x_j$$

$$\sum_{C_j \in \delta^+(S)} x_j \geq 1 \quad \forall S \subseteq V \setminus \{r\}, \ S \cap R \neq 0$$

$$x_e \geq 0 \quad \forall e \in E$$
Algorithm

Iterative Randomized Rounding

- Solve LP-relaxation of the problem
- Select a component, $C_j$, with probability $\frac{x_j}{\sum x_j}$
- Shrink $C_j$ to a single terminal
- Iterate the process on the residual problem
Algorithm
Algorithm

For $t = 1, 2, 3...\mu$
- Compute an optimal fractional solution $x^t$ to $k$-DCR (w.r.t. the current instance)
- Sample a component $C^t$, where $C^t = C_j$ with probability $\frac{x_j^t}{\sum x_i^t}$
- Contract $C^t$

Compute a terminal spanning tree $T^{\mu}$ on the remaining instance

Output $T^{\mu} \cup \bigcup_{t=1}^{\mu} C^t$
Choose a component $C_j$ with $x_j > 0$

Double its edges to form a Eulerian tour

Cost of Eulerian tour $\leq 2 \cdot c_j$
\[ C(T) \leq 2\text{opt}_k^f \]

- Shortcut Eulerian tour to a TSP tour
- Delete one edge of TSP tour to make terminal spanning tree \( T_j \)
- Cost \( T_j \leq 2 \cdot c_j \)
\[ C(T) \leq 2^{opt_k^f} \]
$C(T) \leq 2opt_k^f$
\[ C(T) \leq 2\text{opt}_k^f \]
Bridges

$Br_T(C_j)$ := bridges of tree $T$ with respect to $C_j$

$br_T(C_j)$ := cost of edges in $Br_T(C_j)$

Finding $Br_T(C_j)$:

- contract $C_j$ into a single vertex
- find minimum spanning subtree (of $T$) on new graph
- the bridges are all of the edges not in the subtree
Let $T$ be a terminal spanning tree and $C_j$ be one component.
Reflecting the bridges
Theorem (Bridge Lemma)

Let $T$ be a terminal spanning tree and $x$ be a $k$-DCR solution. Then

$$c(T) \leq \sum_j x_j br_T(C_j)$$
Statement of Bridge Lemma

Terminal spanning tree $T$
Statement of Bridge Lemma

LP solution and components $C_j$ with $x_j > 0$
Statement of Bridge Lemma

Find $Br_T(C_j)$ and reflect them
Statement of Bridge Lemma

Find $Br_T(C_j)$ and reflect them
New terminal spanning tree $Y$

$w(e)$ in $Y$ is inherited from bridge weights

Any spanning tree on $Y$ costs more than $T$
$x_2 + x_3 \geq 1$
Statement of Bridge Lemma

- \( y(e) = \sum_{e \in C_j} x_j \)
- \( y(e) \)'s are feasible solution for BCR
Bridge Lemma

**Theorem (Edmonds)**

If \( R = V \) \( \Rightarrow \) BCR is integral

There exist an integral terminal spanning tree \( F \) s.t.

\[
w(F) \leq \sum_{e \in Y} w(e)y(e)
\]

\[
\sum_j x_j br_T(Cj) = \sum_j x_j w(Yj) = \sum_{e \in Y} w(e)y(e) \geq w(F) \geq c(T)
\]
Bridge Lemma

Let $T$ be a terminal spanning tree and $x$ be a $k$-DCR solution. Then

$$c(T) \leq \sum_j x_j br_T(C_j)$$
Bound on Bridges Weight

\begin{figure}
\centering
\begin{tikzpicture}
  \node[fill] (v) at (0,0) {}; 
  \node (a) at (-1,1) {}; 
  \node (b) at (-1,-1) {}; 
  \node (c) at (1,1) {}; 
  \node (d) at (1,-1) {}; 
  \draw (a) -- (v) node[midway,above] {1} -- (b); 
  \draw (c) -- (v) node[midway,above] {1} -- (d); 
  \draw (v) -- (a) node[midway,right] {2}; 
  \draw (v) -- (b) node[midway,right] {2}; 
  \draw (v) -- (c) node[midway,right] {3}; 
  \draw (v) -- (d) node[midway,right] {3}; 
\end{tikzpicture}
\end{figure}
Bound on Bridges Weight
Bound on Bridges Weight
Algorithm

For $t = 1, 2, 3...\mu$

- Compute an optimal fractional solution $x^t$ to $k$-DCR (w.r.t. the current instance)
- Sample a component $C^t$, where $C^t = C_j$ with probability $\frac{x_j^t}{\sum x_i^t}$
- Contract $C^t$

Compute a terminal spanning tree $T^\mu$ on the remaining instance

Output $T^\mu \cup \bigcup_{t=1}^{\mu} C^t$
Slightly different LP
- $\sum x_j$ might vary through iterations.
- Add a dummy component $C$ containing only $r$
- $c(C) = 0$
- Add the constraint $\sum x_j = \Sigma$
- $\Sigma$ is an upper bound on $\sum x_j$
Improved Approximation for Steiner Tree

New LP for $k$-DCR:

$$\begin{align*}
\min & \sum_j c(C_j)x_j \\
\sum_{C_j \in \delta^+(S)} x_j & \geq 1 & \forall S \subseteq V - r \\
x_e & \geq 0 & \forall e \in E \\
\sum_j x_j & = \Sigma
\end{align*}$$
Bound $T^\mu$

Lemma

\[ E[c(T^\mu)] \leq 2 \left( 1 - \frac{1}{\sum} \right)^\mu opt_k^f \]
Bound $T^\mu$

\[ E[c(T^t)] \leq c(T^{t-1}) - E[br_T^{t-1}(C^t)] \]
Bound $T^\mu$

MST on terminals and potential component
Bound $T^\mu$

Bridges for this component on tree T
Bound $T^\mu$

- Bridges form cycles in new MST
- Delete bridges we have new MST
Bound $T^\mu$

Lemma

$$E[c(T^\mu)] \leq 2\left(1 - \frac{1}{\Sigma}\right)^\mu opt_k^f$$

$$E[c(T^t)] \leq c(T^{t-1}) - E[br_T^{t-1}(C^t)]$$

$$= c(T^{t-1}) - \frac{1}{\Sigma} \sum_j x_j^t br_T^{t-1}(C_j)$$

$$\leq (1 - \frac{1}{\Sigma})c(T^{t-1})$$ (Bridge Lemma)

$$E[c(T^\mu)] \leq (1 - \frac{1}{\Sigma})^\mu c(T^0) \leq 2\left(1 - \frac{1}{\Sigma}\right)^\mu opt_k^f$$
Lemma

Let $S^t$ be the optimal steiner tree after $t$ iterations, then we have

$$E[c(S^t)] \leq (1 - \frac{1}{2\Sigma})^t c(S)$$
\[ E[c(S^t)] = c(S^{t-1}) - E[c(C_j)] \leq c(S^{t-1}) - E[br_{S^{t-1}}(C_j)] \]
Bound Component

Reflect $Y$ tree
Bound Component
Bound Component

- Bridges based on Steiner tree
- Bridges based on $Y$ tree
Lemma

Let $S^t$ be the optimal steiner tree after $t$ iterations, then we have

$$E[c(S^t)] \leq (1 - \frac{1}{2\Sigma})^t c(S)$$
\[ E[c(S^t)] = c(S^{t-1}) - E[c(C_j)] \]
\[ \leq c(S^{t-1}) - E[br_{S^{t-1}}(C_j)] \]
\[ \leq c(S^{t-1}) - E[br_Y(C_j)] \]
\[ = c(S^{t-1}) - \frac{1}{\sum} \sum_{j} x_j br_Y(C_j) \]
\[ \leq c(S^{t-1}) - \frac{1}{\sum} w(Y) \]
\[ \leq c(S^{t-1}) - \frac{1}{2\sum} c(S^{t-1}) \]
\[ = (1 - \frac{1}{\sum}) c(S^{t-1}) \]
Lemma

Final $c(S^t)$ relates to steiner tree before first iteration

$$E[c(S^t)] \leq (1 - \frac{1}{2\Sigma})^{t-1}c(S)$$
Corollary

For every $t = 1, ..., \mu$, $E[c(C^t)] \leq \frac{1}{\sum}(1 - \frac{1}{2\sum})^{t-1} opt_k$

\[
E[c(C^t)] \leq \frac{1}{\sum} E\left[\sum_j (x_j)^t c(C_j)\right]
\]

\[
= \frac{1}{\sum} E[(opt_k)^{f,t}]
\]

\[
= \frac{1}{\sum} E[(opt_k)^t]
\]

\[
\leq (1 - \frac{1}{2\sum})^{t-1} opt_k
\]
Lemma

For any $k = O(1)$, there is a polynomial-time expected $\frac{1}{2}$-approximation algorithm for $k$-restricted Steiner tree

\[
E \left[ \frac{c(T^\mu)}{c(S)} + \sum_{t=1}^{\mu} \frac{c(C^t)}{c(S)} \right] \leq 2 \left( 1 - \frac{1}{\Sigma} \right)^\mu + \frac{1}{\Sigma} \sum (1 - \frac{1}{2\Sigma})^{t-1}
\]

\[
= 2 \left( 1 - \frac{1}{\Sigma} \right)^\delta \Sigma + 2 - 2 \left( 1 - \frac{1}{2\Sigma} \right)^\delta \Sigma
\]

\[
\leq 2e^{-\delta} + 2 - 2e^{-\frac{\delta}{2}}
\]

\[
= \frac{3}{2}
\]