Example. Show that for all integers $n \geq 0$, if $r \neq 1$,

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

Solution. Let $r$ be any real number that is not equal to 1. We want to prove that $\forall$ integers $n, P(n)$, where $P(n)$ is given by

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

Base Case: We want to show that $P(0)$ is true.

$$\sum_{i=0}^{0} ar^i = a = \frac{a(r - 1)}{r - 1}$$

Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.

Induction Step: We want to show that $P(k + 1)$ is true, i.e., we want to prove that

$$\sum_{i=0}^{k+1} ar^i = \frac{a(r^{k+2} - 1)}{r - 1}$$

We can do this as follows.

L.H.S. $= \sum_{i=0}^{k+1} ar^i$

$= \sum_{i=0}^{k} ar^i + ar^{k+1}$

$= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}$

$= \frac{a(r^{k+1} - 1)}{r - 1} + \frac{ar^{k+1}(r - 1)}{r - 1}$

$= \frac{a}{r - 1} \left( r^{k+1} + (r - 1) - 1 \right)$

$= \frac{a}{r - 1} \left( r^{k+2} - 1 \right)$

$= \frac{a(r^{k+2} - 1)}{r - 1}$
Example. Prove that $\forall$ non-negative integers $n$,

$$
\sum_{i=0}^{n} 2^i = 2^{n+1} - 1
$$

Solution. By setting $a = 1, r = 2$ in the result of the previous problem, the claim follows.

Example. Prove that $\forall$ non-negative integers $n$, $2^{2n} - 1$ is a multiple of 3.

Solution. We want to prove that $\forall$ non-negative integers $n$, $P(n)$, where $P(n)$ is

$$
2^{2n} - 1 = 3k, \text{ for some non-negative integer } k
$$

Base Step: $P(0)$ is true as shown below.

$$
2^0 - 1 = 0 = 3 \cdot 0.
$$

Induction Hypothesis: Assume that $P(x)$ is true for some $x \geq 0$, i.e., $2^{2x} - 1 = 3 \cdot k'$, for some $k' \geq 0$.

Induction Step: We want to prove that $P(x + 1)$ is true, i.e., we want to show that

$$
2^{2(x+1)} - 1 = 3l, \text{ for some non-negative integer } l.
$$

We can show this as follows.

\[
\text{L.H.S.} &= 2^{2(x+1)} - 1 \\
&= 2^{2x+2} - 1 \\
&= 2^{2x} \cdot 2^2 - 1 \\
&= 2^{2x} \cdot 4 - 1 \\
&= 2^{2x} \cdot (3 + 1) - 1 \\
&= 3 \cdot 2^{2x} + 2^{2x} - 1 \\
&= 3 \cdot 2^{2x} + 3 \cdot k' \quad \text{(using induction hypothesis)} \\
&= 3(2^{2x} + k') \\
&= 3l, \quad \text{where } l = 2^{2x} + k'
\]

Since $x$ and $k'$ are integers $l$ is also an integer. Hence, $P(x + 1)$ is true.

Example. Prove that $\forall n \in \mathbb{N}, n > 1 \rightarrow n! < n^n$.

Solution. Below is a simple direct proof (as suggested in the class by Tim Frick) for this inequality.

\[
n! = 1 \times 2 \times 3 \times \cdots \times n \\
< n \times n \times n \times \cdots \times n \\
= n^n
\]
We now give a proof using induction. Let $P(n)$ denote the following property.

$$n! < n^n$$

**Base Case:** We want to prove $P(2)$. $P(2)$ is the proposition that $2! < 2^2$, or $2 < 4$, which is true.

**Induction Hypothesis:** Assume that $P(k)$ is true for some $k > 1$.

**Induction Step:** We want to prove $P(k+1)$, i.e., we want to prove that $(k+1)! < (k+1)^{k+1}$.

L.H.S. = $(k + 1)!$

$$= k! \times (k + 1)$$

$$< k^k \times (k + 1)$$

(using induction hypothesis)

$$< (k + 1)^k \times (k + 1)$$

(since $k > 1$)

$$= (k + 1)^{k+1}$$

**Example.** Let $n$ be a non-negative integer. Show that any $2^n \times 2^n$ region with one central square removed can be tiled using L-shaped pieces, where the pieces cover three squares at a time (Figure 1).

**Solution.** (Attempt 1) Let $R_n$ denote a $2^n \times 2^n$ region. Let $P(n)$ be the property that $R_n$ with one central square removed can be tiled using L-shaped pieces.

Figure 1: A L-tile and an L-tiling of a $2^2 \times 2^2$ region without a square.

**Base Case:** We want to prove that $P(0)$ is true. This is true because a $1 \times 1$ region with one central square removed requires 0 tiles.

**Induction Hypothesis:** Assume that $P(k)$ is true for some $k > 0$.

**Induction Step:** We want to prove that $P(k+1)$ is true, i.e., region $R_{k+1}$ with one central square removed can be tiled using L-shaped pieces.

$R_{k+1}$ can be divided into four regions of size $2^k \times 2^k$. Note that the four central corners of $R_{k+1}$ can be covered using one L-shaped tile and one square hole (Figure 2). Each of the four remaining regions has one hole and is of the size $2^k \times 2^k$. By induction hypothesis, these regions can be covered using L-shaped pieces. Thus, since the four disjoint regions
can be covered using L-shaped tiles, $R_{k+1}$ without a central square can also be covered using L-shaped tiles.

![Illustration of the two proof attempts.](image)

Our use of induction hypothesis is incorrect as we have assumed that region $R_k$ without a *central* square (not a *corner* square) can be covered using L-shaped tiles.

Surprisingly, we can get around this obstacle by proving the following stronger claim.

“For all positive integers $n$, any $R_n$ region with *any* one square removed can be L-tiled.”

Let $P(n)$ be the property that $R_n$ without one square can be L-tiled.

**Base Case:** We want to prove that $P(0)$ is true. This is true because a $1 \times 1$ region with one square removed requires 0 tiles.

**Induction Hypothesis:** Assume that $P(k)$ is true for some $k$.

**Induction Step:** We want to prove that $P(k+1)$ is true, i.e., region $R_{k+1}$ without one square that is located anywhere can be L-tiled. Divide $R_{k+1}$ into four $R_k$ regions. One of the four $R_k$ regions that does not have one square can be L-tiled (using induction hypothesis). Each of the other three $R_k$ regions without the corner square that is located at the center of $R_{k+1}$ can be L-tiled (using induction hypothesis). By using one more L-tile we can cover the three central squares of $R_{k+1}$.