Example. Prove that $\sqrt{2}$ is irrational.

Solution. For the purpose of contradiction, assume that $\sqrt{2}$ be a rational number. Then there are numbers $a$ and $b$ with no common factors such that

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides of the above equation gives

$$2 = \frac{a^2}{b^2}$$
$$a^2 = 2b^2$$

(1)

From (1) we conclude that $a^2$ is even. This fact combined with the result of Example 1 implies that $a$ is even. Then, for some integer $k$, let

$$a = 2k$$

(2)

Combining (1) and (2) we get

$$4k^2 = 2b^2$$
$$2k^2 = b^2$$

The above equation implies that $b^2$ is even and hence $b$ is even. Since we know $a$ is even this means that $a$ and $b$ have 2 as a common factor which contradicts the assumption that $a$ and $b$ have no common factors.

We will now give a very elegant proof for the fact that “$\sqrt{2}$ is irrational” using the unique factorization theorem which is also called the fundamental theorem of arithmetic.

The unique factorization theorem states that every positive number can be uniquely represented as a product of primes. More formally, it can be stated as follows.

Given any integer $n > 1$, there exist a positive integer $k$, distinct prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$ such that

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}$$

and any other expression of $n$ as a product of primes is identical to this except, perhaps, for the order in which the factors are written.
Example. Prove that $\sqrt{2}$ is irrational using the unique factorization theorem.

Solution. Assume for the purpose of contradiction that $\sqrt{2}$ is rational. Then there are numbers $a$ and $b$ ($b \neq 0$) such that

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides of the above equation gives

$$2 = \frac{a^2}{b^2}$$

$$a^2 = 2b^2$$

Let $S(m)$ be the sum of the number of times each prime factor occurs in the unique factorization of $m$. Note that $S(a^2)$ and $S(b^2)$ is even. Why? Because the number of times that each prime factor appears in the prime factorization of $a^2$ and $b^2$ is exactly twice the number of times that it appears in the prime factorization of $a$ and $b$. Then, $S(2b^2)$ must be odd. This is a contradiction as $S(a^2)$ is even and the prime factorization of a positive integer is unique.

Example. Prove or disprove that the sum of two irrational numbers is irrational.

Solution. The above statement is false. Consider the two irrational numbers, $\sqrt{2}$ and $-\sqrt{2}$. Their sum is $0 = 0/1$, a rational number.

Example. Show that there exist irrational numbers $x$ and $y$ such that $x^y$ is rational.

Solution. We know that $\sqrt{2}$ is an irrational number. Consider $\sqrt{2}^{\sqrt{2}}$.

Case I: $\sqrt{2}^{\sqrt{2}}$ is rational.
In this case we are done by setting $x = y = \sqrt{2}$.

Case II: $\sqrt{2}^{\sqrt{2}}$ is irrational.
In this case, let $x = \sqrt{2}^{\sqrt{2}}$ and let $y = \sqrt{2}$. Then, $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^2 = 2$, which is an integer and hence rational.

Example. Prove that for all positive integers $n$,

$$n \text{ is even } \iff 7n + 4 \text{ is even}$$

Solution. Let $n$ be a particular but arbitrarily chosen integer.

Proof for $n$ is even $\rightarrow$ $7n + 4$ is even. Since $n$ is even, $n = 2k$ for some integer $k$. Then,

$$7n + 4 = 7(2k) + 4 = 2(7k + 2)$$
Hence, \(7n + 4\) is even.

**Proof for \(7n + 4\) is even \(\rightarrow\) \(n\) is even.** Since \(7n + 4\) is even, \(7n + 4 = 2l\) for some integer \(l\). Then,

\[
7n = 2l - 4 = 2(l - 2)
\]

Clearly, \(7n\) is even. Combining the fact that \(7\) is odd with the result of the Example 1, we conclude that \(n\) is even.

We can also prove the latter by proving its contrapositive, i.e., we can prove

if \(n\) is odd then \(7n + 4\) is odd.

Since \(n\) is odd we have \(n = 2k + 1\), for some integer \(k\). Thus we have

\[
7n + 4 = 7(2k + 1) + 4 = 14k + 10 + 1 = 2(7k + 5) + 1 = 2k' + 1, \text{ where } k' = 7k + 5 \text{ is an integer.}
\]

**Example.** Prove that there are infinitely many prime numbers.

**Solution.** Assume, for the sake of contradiction, that there are only finitely many primes. Let \(p\) be the largest prime number. Then all the prime numbers can be listed as

\[2, 3, 5, 7, 11, 13, \ldots, p\]

Consider an integer \(n\) that is formed by multiplying all the prime numbers and then adding 1. That is,

\[n = (2 \times 3 \times 5 \times 7 \times \cdots \times p) + 1\]

Clearly, \(n > p\). Since \(p\) is the largest prime number, \(n\) cannot be a prime number. In other words, \(n\) is composite. Let \(q\) be any prime number. Because of the way \(n\) is constructed, when \(n\) is divided by \(q\) the remainder is 1. That is, \(n\) is not a multiple of \(q\). This contradicts the Fundamental Theorem of Arithmetic.

**Alternate Proof by Filip Saidak.** Let \(n\) be an arbitrary positive integer greater than 1. Since \(n\) and \(n + 1\) are consecutive integers, they must be relatively prime. Hence, the number \(N_2 = n(n + 1)\) must have at least two different prime factors. Similarly, since the integers \(n(n + 1)\) and \(n(n + 1) + 1\) are consecutive, and therefore relatively prime, the number

\[N_3 = n(n + 1)[n(n + 1) + 1]\]

must have at least three different prime factors. This process can be continued indefinitely, so the number of primes must be infinite.
Mathematical Induction

Example. Prove that the sum of the first $n$ positive odd numbers is $n^2$.

Solution. We want to prove that $\forall$ positive integers $n, P(n)$ where $P(n)$ is the following property.

$$\sum_{i=0}^{n-1} 2i + 1 = n^2$$

Base Case: We want to show that $P(1)$ is true. This is clearly true as

$$\sum_{i=0}^{0} 2i + 1 = 1 = 1^2$$

Induction Hypothesis: Assume $P(k)$ is true for some $k \geq 1$.

Induction Step: We want to show that $P(k + 1)$ is true, i.e., we want to show that

$$\sum_{i=0}^{k} 2i + 1 = (k + 1)^2$$

We can do this as follows.

$$\sum_{i=0}^{k} 2i + 1 = \sum_{i=0}^{k-1} 2i + 1 + 2k + 1 = k^2 + 2k + 1 \quad \text{(using induction hypothesis)}$$

$$= (k + 1)^2$$