Example. Prove that \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).

Solution. We pose the following counting question.

Given a set \( S \) of \( n \) distinct elements how many subsets are there of the set \( S \)?

From Lecture 2 (Example 1) we know that the answer is \( 2^n \). This gives us the RHS.

Another way to compute the answer to the question is as follows. The set \( P(S) \) of all possible subsets can be partitioned into \( S_0, S_1, \ldots, S_n \), where \( S_i, 0 \leq i \leq n \), is the set of all subsets of \( S \) that have cardinality \( i \). Thus

\[
|P(S)| = |S_0| + |S_1| + \ldots + |S_n|
= \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n}
= \sum_{k=0}^{n} \binom{n}{k} = \text{LHS}
\]

This proves the claim.

Example. Prove that \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \).

Solution. We pose the following counting question.

How many ways are there to choose two numbers from \( S = \{0, 1, 2, \ldots, n\} \)?

By definition, there are \( \binom{n+1}{2} = \frac{n(n+1)}{2} \) distinct pairs of \( S \). This gives us the RHS.

We can also compute the answer as follows. Let \( P \) be the set of all pairs of \( S \). \( P \) can be partitioned into \( S_1, S_2, \ldots, S_n \), where \( S_i, 1 \leq i \leq n \), is the set of pairs in which \( i \) is the bigger element in the pair. Clearly,

\[
|P| = |S_1| + |S_2| + \ldots + |S_n|
= 1 + 2 + \ldots + n
= \sum_{k=1}^{n} k = \text{LHS}
\]

This proves the claim.
Functions

Definition. Let \( A \) and \( B \) be sets. A function \( f \) is a mapping from \( A \) to \( B \) such that each element in \( A \) is mapped to exactly one element in \( B \). We write \( f(x) = y \) where \( y \in B \) is the unique element that \( a \in A \) is mapped to. We denote such a function by \( f : A \to B \).

Definition. Given a function \( f : A \to B \), \( A \) is called the domain of \( f \) and \( B \) is called the co-domain of \( f \). If \( f(x) = y \) then \( y \) is called the image of \( x \) and \( x \) is called the preimage of \( y \). The range of \( f \) is the set of all images of elements in \( A \).

Definition. A function \( f \) is a one-to-one function if \( f(x) = f(y) \) implies \( x = y \). For example, the function \( f(x) = x^2 \) from the set of integers to the set of integers is not one-to-one because \( f(x_1) = f(x_2) \) when \( x_1 = -2 \) and \( x_2 = 2 \), however \( x_1 \neq x_2 \). On the other hand the function \( f : \mathbb{R} \to \mathbb{R} \), defined by the rule \( f(x) = 4x - 1 \), for all real numbers \( x \), is one-to-one.

Definition. A function \( f \) is an onto function if its range equals its co-domain. For example, the function \( f(x) = x + 1 \) from the set of integers to the set of integers is an onto function.

Definition. A function \( f \) is a one-to-one correspondence, if it is both one-to-one and onto.

The Pigeonhole Principle

If \( k + 1 \) or more objects are distributed among \( k \) bins then there is at least one bin that has two or more objects. For example, the pigeon hole principle can be used to conclude that in any group of thirteen people there are at least two who are born in the same month.

Example. There are \( n \) pairs of socks. How many socks must you pick without looking to ensure that you have at least one matches pair?

Solution. The pigeonhole principle can be applied by letting \( n \) bins correspond to the \( n \) pairs of socks. If we select \( n + 1 \) socks and put each one in the box corresponding to the pair it belongs to then there must be at least one box containing a matched pair.

The Generalized Pigeonhole Principle

If \( n \) objects are placed into \( k \) boxes, then there is at least one box containing at least \( \lceil n/k \rceil \) objects.
Proof: Assume otherwise, i.e., each box contains at most \( \lceil n/k \rceil - 1 \) objects. Then, the total number of objects is at most

\[
k \left( \lceil \frac{n}{k} \rceil - 1 \right) < k \left( \frac{n}{k} + 1 - 1 \right) = n
\]

This is a contradiction as there are \( n \) objects.

Using the generalized pigeonhole principle we can conclude that among 100 people, there are at least \( \lceil 100/12 \rceil = 9 \) who are born in the same month.

Example. Suppose each point in the plane is colored either red or blue. Show that there always exist two points of the same color that are exactly one feet apart.

Solution. Consider an equilateral triangle with the length of each side being one feet. The three corners of the triangle are colored red or blue. By pigeonhole principle, two of these three points must have the same color.

Example. Given a sequence of \( n \) integers, show that there exists a subsequence of consecutive integers whose sum is a multiple of \( n \).

Solution. Let \( x_1, x_2, \ldots, x_n \) be the sequence of \( n \) integers. Consider the following \( n \) sums.

\[
x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \cdots + x_n
\]

If any of these \( n \) sums is divisible by \( n \), then we are done. Otherwise, each of the \( n \) sums have a non-zero remainder when divided by \( n \). There are at most \( n - 1 \) different possible remainders: \( 1, 2, \ldots, n - 1 \). Since there are \( n \) sums, by the pigeonhole principle, at least two of the \( n \) sums have the same remainder when divided by \( n \). Let \( p \) and \( q, p < q \), be integers such that for some integers \( c_1 \) and \( c_2 \),

\[
x_1 + x_2 + \cdots + x_p = c_1 n + r \quad \text{and} \quad x_1 + x_2 + \cdots + x_q = c_2 n + r
\]

Subtracting the two sums, we get

\[
x_{p+1} + \cdots + x_q = (c_2 - c_1)n
\]

Hence, \( x_{p+1} + \cdots + x_q \) is divisible by \( n \).

Example. Show that in any group of six people there are either three mutual friends or three mutual strangers.

Solution. Consider one of the six people, say \( A \). The remaining five people are either friends of \( A \) or they do not know \( A \). By the pigeonhole principle, at least \( \lceil 5/2 \rceil = 3 \) of the five people are either friends of \( A \) or are unacquainted with \( A \). In the former case, if any two of the three people are friends then these two along with \( A \) would be mutual friends, otherwise the three people would be strangers to each other. The proof for the latter case, when three or more people are unacquainted with \( A \), proceeds in the same manner.
Example. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists consecutive days during which the chess master will have played exactly 21 games.

Solution. Let $a_i$, $1 \leq i \leq 77$, be the total number of games that the chess master has played during the first $i$ days. Note that the sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence. We have

$$1 \leq a_1 < a_2 < \ldots < a_{77} \leq 11 \times 12 = 132$$

Now consider the sequence $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$. We have

$$22 \leq a_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \leq 153$$

Clearly, this sequence is also a strictly increasing sequence. The numbers $a_1, a_2, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ (154 in all) belong to the set $\{1, 2, \ldots, 153\}$. By the pigeonhole principle there must be two numbers out of the 154 numbers that must be the same. Since no two numbers in $a_1, a_2, \ldots, a_{77}$ are equal and no two numbers in $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are equal there must exist $i$ and $j$ such that $a_i = a_j + 21$. Hence during the days $j + 1, j + 2, \ldots, i$, exactly 21 games must have been played.