The Inclusion-Exclusion Formula.

If $A, B,$ and $C$ are any finite sets, then

\[
|A \cup B| = |A| + |B| - |A \cap B|
\]

\[
|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C|
\]

Observe that if the sets $A, B,$ and $C$ are mutually disjoint, i.e., $A \cap B = A \cap C = B \cap C = \emptyset$ then we get

\[
|A \cup B| = |A| + |B|
\]

\[
|A \cup B \cup C| = |A| + |B| + |C|
\]

This is often called the *addition rule* or the *sum rule*.

**Example.** In how many ways can we select two books from different subjects among five distinct computer science books, three distinct math books, and two distinct art books?

**Solution.** The set of all possible two books from different subjects can be partitioned into three subsets, $S_1, S_2,$ and $S_3$. The subset $S_1$ contains two books belonging to computer science and math, the subset $S_2$ contains two books belonging to computer science and art, and the subset $S_3$ contains two books belonging to math and art. We have

\[
|S_1| = 5 \times 3 = 15
\]

\[
|S_2| = 5 \times 2 = 10
\]

\[
|S_3| = 3 \times 2 = 6
\]

By the addition rule, total number of ways of selecting 2 books from different subjects equals $|S_1| + |S_2| + |S_3| = 31$.

**Example.** A PIN is typically made of four symbols chosen from 26 letters of the alphabet and the 10 digits, with repetitions allowed. How many PINS contain repeated symbols?
Solution. Let $S$ denote the set of all possible PINs of four alpha-numeric characters. Let $S_1$ denote the set of all possible PINs of four alpha-numeric characters with no repeated symbols. Let $S_2$ denote the set of all possible PINs of four alpha-numeric characters with some symbols repeated. By the addition rule,

$$|S| = |S_1| + |S_2|$$

By simple application of multiplication rule, we see that $|S| = 36^4 = 1679616$ and $|S_1| = 36 \times 35 \times 34 \times 33 = 1413720$. Plugging these values in the above equation, we get $|S_2| = 265896$.

Example. (a) How many integers from 1 through 1000 are multiples of 3 or multiples of 5?
(b) How many integers from 1 through 1000 are neither multiples of 3 nor multiples of 5?

Solution. (a) Let $S = \{1, 2, 3, \ldots, 1000\}$. Let $M \subseteq S$ be the set of integers that are multiples of 3 or multiples of 5. Let $M_1 \subseteq S$ be the set of integers that are multiples of 3. Let $M_2 \subseteq S$ be the set of integers that are multiples of 5. Note that the first integer in $S$ that is divisible by 3 is 3 = 3 \times 1. The last integer in $S$ that is divisible by 3 is 999 = 3 \times 333. Thus, $|M_1| = 333$. Similarly, $|M_2| = 200$. Note that $M_1$ and $M_2$ are not disjoint, i.e., there are integers like 15 that are divisible by 3 and by 5 and hence exist in $M_1$ as well as $M_2$. We have double-counted them. So now, let’s find the size of the set $M_1 \cap M_2$. Observe that each element in $M_1 \cap M_2$ must be a multiple of $3 \times 5 = 15$. The first number in $S$ that is a multiple of 15 is 15 = 15 \times 1 and the last number in $S$ that is a multiple of 15 is 990 = 15 \times 66. Thus, $|M_1 \cap M_2| = 66$. By the inclusion-exclusion formula, we get

$$|M| = |M_1| + |M_2| - |M_1 \cap M_2| = 333 + 200 - 66 = 467$$

(b) Let $N \subseteq S$ be the set of integers that are neither multiples of 3 nor multiples of 5. Note that the sets $M$ and $N$ form a partition of the set $S$. Applying the addition rule we get

$$|S| = |M| + |N|$$

Thus,

$$|N| = |S| - |M| = 1000 - 467 = 533$$

Combinations.

Let $n$ and $r$ be non-negative integers. An $r$-combination of a set of $n$ elements means an unordered selection of $r$ of the $n$ elements of $S$. The symbol $\binom{n}{r}$ (read as “$n$ choose $r$”) denotes the number of $r$-combinations of a set of $n$ elements. This is same as the number of subsets of size $r$ that can be chosen from a set of $n$ elements.
The following numbers can be verified easily.

\[
\binom{n}{r} = \begin{cases} 
0 & \text{if } r > n \\
1 & \text{if } r = 0 \text{ or } r = n \\
n & \text{if } r = 1
\end{cases}
\]

Do you see the distinction between a \(r\)-permutation and a \(r\)-combination? A \(r\)-permutation is an ordered selection of \(r\) elements, i.e., both, which \(r\) elements, as well as the order in which they are chosen are important. Two \(r\)-permutations are the same if the \(r\) elements chosen are the same and they are chosen in the same order. In contrast, in a \(r\)-combination, only the choice of \(r\) elements is important. The order in which the \(r\) elements are chosen is irrelevant. Two \(r\)-combinations are the same if they have the same \(r\) elements regardless of the orders of selection of these elements.

In general, what is the value of \(\binom{n}{r}\), i.e., how many \(r\)-combinations are possible if we have a set of \(n\) distinct objects?

We will answer this question by giving an expression that relates \(\binom{n}{r}\) and \(P(n, r)\). A \(r\)-permutation can be obtained in two steps as follows.

1. **Step 1.** Choose \(r\) elements from the available \(n\) elements.
2. **Step 2.** Arrange the chosen \(r\) elements.

Step 1 can be performed in \(\binom{n}{r}\) ways. Step 2 can be performed in \(r!\) ways. By the multiplication rule, the total number of \(r\)-permutations is given by

\[
P(n, r) = \binom{n}{r} \times r!
\]

Rearranging the terms of the above equation we get

\[
\binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}
\]

**Example.** We have a pool of 14 players from which 11 players must be chosen to play a cricket match? How many 11-member teams are possible?

**Solution.** The number of distinct 11-member teams is the same as the number of subsets of size 11 from the set of 14 players. This is given by

\[
\binom{14}{11} = \frac{14!}{11!3!} = \frac{12 \times 13 \times 14}{1 \times 2 \times 3} = 364.
\]

**Example.** Consider a set of twenty-five points, no three of which are collinear. How many straight lines do they determine? How many triangles do they determine?
Solution. Since no three points lie on a straight line, every two points determine a straight line. The number of straight lines equals the number of 2-combinations of a 25-element set. This is given by
\[
\binom{25}{2} = \frac{25!}{2!23!} = \frac{24 \times 25}{1 \times 2} = 300.
\]
Similarly every three points determine a triangle. Thus the number of triangles is given by
\[
\binom{25}{3} = \frac{25!}{3!22!} = \frac{23 \times 24 \times 25}{1 \times 2 \times 3} = 2300.
\]

Example. From a group of 8 women and 6 men, how many different committees consisting of 3 women and 2 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

Solution. The procedure of forming a committee of 3 women and 2 men is as follows.

Step 1. Choose the 3 women.
Step 2. Choose the 2 men

Step 1 can be done in \(\binom{8}{3}\) ways. Step 2 can be done in \(\binom{6}{2}\) ways. Using the multiplication rule, the total number of possible committees is \(\binom{8}{3} \times \binom{6}{2} = 840\).

The second part of the question can be solved as follows. Let \(S_1\) be the set of all possible committees that do not contain the two feuding men. Let \(S_2\) be the set of all possible committees that contain exactly one of the two feuding men. Clearly, the no. of possible committees that do not contain the two feuding men together equals \(|S_1| + |S_2|\).

Using the reasoning used in the first part of the question we get \(|S_1| = \binom{8}{3} \times \binom{4}{2} = 336\) and \(|S_2| = 2 \binom{8}{3} \times \binom{4}{1} = 448\). Hence the total number of committees without the two feuding men together is 336 + 448 = 784.

The answer to the second part could also be derived by finding the number of all possible committees and then subtracting the number of committees in which the two feuding men are together. There are \(\binom{8}{3} \binom{6}{2} = 840\) committees in all out of which \(\binom{8}{3} \binom{2}{1} = 56\) committees contain the two feuding men. Thus there are 840 − 56 = 784 committees in all that have non-feuding men.

Example. There are 15 students enrolled in CS 171, but exactly 12 students attend on any given day. The classroom for CS 171 has 25 distinct seats. How many different classroom seatings are possible?

Solution. A classroom seating can be constructed in two steps as follows.

Step 1. Choose 12 students out of 15 that are enrolled.
Step 2. Arrange 12 students in 25 distinct seats available.

Step 1 can be performed in \(\binom{15}{12}\) ways. Step 2 can be performed in \(P(25, 12)\) ways. By the multiplication rule, the number of different classroom seatings possible is given by
\[
\binom{15}{12} \times P(25, 12) = \frac{15!}{12!3!} \times \frac{25!}{13!}.
\]