1. Prove that for $n \in \mathbb{N}$, with $n \geq 2$, define $s_n$ by

$$s_n = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right).$$

Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

2. Prove that $\sqrt{6}$ is irrational.

3. Prove the following using induction.

   (a) $\forall n \geq 1, \sum_{j=1}^{n} j^2 = n(n + 1)(2n + 1)/6.$
   (b) If $n$ is a positive integer and $1 + x > 0$ then $(1 + x)^n \geq 1 + nx$.
   (c) for all positive integers $n$ and for all integers $x > 1$, $x^n - 1$ is divisible by $x - 1$.

4. Find the mistake in the attempted proof given below of the following ridiculous claim: “All horses are of the same color”.

   Let $P(n)$ be the property that $n$ horses are of the same color.
   
   **Base Case:** $P(1)$ is clearly true since there is only one horse.
   
   **Induction hypothesis:** Assume that $P(k)$ is true for some $k > 0$.
   
   **Induction Step:** We want to prove that $P(k+1)$ is true. Consider any set of $k+1$ horses and number them $1, 2, \ldots, k+1$. By induction hypothesis, the first $k$ horses are of the same color. Also, the last $k$ horses are of the same color, again by induction hypothesis. Since the set of first $k$ horses and the set of last $k$ horses overlap, all $k+1$ horses must be of the same color. Thus, $P(k+1)$ is true. This completes the proof.

5. Suppose that $m$ and $n$ are integers such that $n^2 + 1 = 2m$. Prove that $m$ is the sum of the squares of two non-negative integers.