1. Prove, using induction, that every positive integer can be expressed as a sum of distinct powers of 2. For example, $13 = 2^3 + 2^2 + 2^0$.  

2. Prove the following using induction.
   
   (a) For all positive integers $n$, and for any integers $a$ and $b$ with $a \neq b$, $a^n - b^n$ is divisible by $a - b$.  
   
   (b) For all positive integers $n$, 
   
   $$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$  

3. Suppose you take a piece of paper and draw a bunch of straight lines, no one exactly on top of another, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only two colors, so that any two regions that share a boundary line have different colors. Regions that share only a boundary point are permitted to have the same color.  

4. Prove or disprove the following. In any group of two or more people, there are always at least two people who have the same number of friends. Assume that if a person $p$ is a friend of a person $q$ then $q$ is also a friend of $p$.  