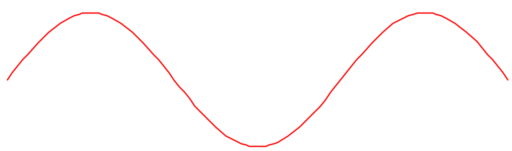
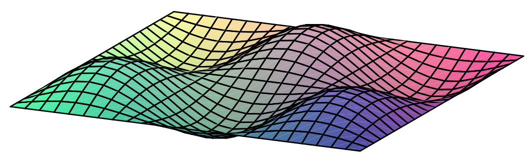
Classical Wave Equation

The purpose of this Lab is to examine the vibrations of a violin string. We'll also have a look at the vibrations of a rectangular drum, and the Maple graphics package will be used to visualize the vibrations.

For this Lab a hand-written report is O.K.

Theory

(Ref: McQuarrie and Simon, *Physical Chemistry*, Ch 2)

The **classical wave equation** for a string is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

where $u = u(x,t)$ and v is the string's velocity.

The ***stationary* states**, also known as harmonics or overtones, for a string of length L are,

$$u_n(x,t) = X_n(x)\,T_n(t) \qquad n = 1,2,3\ldots$$

where,

$$X_n(x) = A_n\,sin(n\pi\,x/L) \qquad (A_n \text{ is the amplitude})$$

and,

$$T_n(t) = cos(n\pi\,v\,t/L + \phi_n) \qquad (\phi_n \text{ is the phase lag})$$

The general solution is a ***travelling* wave**,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n\,sin(n\pi\,x/L)\,cos(n\pi\,v\,t/L+\phi_n)$$

For a **square membrane (drum)** of sidelength L, the stationary states are,

$$u_{m,n}(x,y,t) = X_m(x)X_n(y)T_{m,n}(t) \qquad m = 1,2,3\ldots, \quad n = 1,2,3\ldots$$

where X_m and X_n are as above (though X_n is now a function of y rather than x) and,

$$T_{m,n}(t) = cos((m+n)\pi\,v\,t/L + \phi_{m,n})$$

In this Lab, we will set $L = 1$, $v = 1$, $\phi_n = 0$, and we will also set most of the A_n's to be zero, eg $A_1 = 1, A_2 = 1, A_n = 0, n \geq 3$.

Using Maple

Getting started

To start maple,

xmaple &

In the maple window,

with(plots);

(This loads the plotting/graphics package).

Defining functions

To define the function $X_1(x)$,

X1 := x → sin(Pi*x);

$T_1(t)$ and $u_1(x,t)$ can be defined similarly,

T1 := t → sin(Pi*t);

u1 := (x,t) → X1(x)*T1(t);

When defining $X_2(x)$, note the '$*$' between '2' and 'Pi',

X2 := x → sin(2*Pi*x);

T_2 and u_2 can be defined analogously.

Graphics and animations

To make an animated plot of $u_1(x,t)$ and $u_2(x,t)$,

animate(plot,[{u1(x,t),u2(x,t)},x=0..1],t=0..2, frames=200);

Clicking on the diagram, a "play" button appears in the menu at the top of the page. Click elsewhere and the play button disappears.

Maple crashes frequently, so save your file regularly!

3-D Graphics and animations

To make a three-dimensional animated plot of $u_{1,2}(x,y,t)$,

animate(plot3d,[u12(x,y,t),x=0..1,y=0..1],t=0..2, frames=200);

You can rotate the figure by left-clicking on the figure. Right-clicking opens a menu with various options, such as adding a box/axes.

Saving and printing graphics

The least painful way to *print* maple diagrams is to first export each diagram as a jpg file by right-clicking on the image.

Once the jpg file is saved, you can either email it to yourself or print it.

To print it, in a terminal/console window open the jpg file in the xv graphics program by typing,

xv filename.jpg &

Right-clicking on the image produced by the xv program opens a menu with various options including print.

Problems

1. Sketch the stationary waves $X_1(x)$ to $X_4(x)$ in your lab book.

   Since the string is of length $L = 1$, the x-axis should range from 0 to 1. (You may use maple as an aid if you wish).

   What are the wavelengths of X_1 to X_4? What does X_0 look like?

2. Sketch the time-evolution of the stationary wave $u_1(x,t)$ over one complete period (cycle) of u_1. (Recall that the period is the time it takes the string to return to its initial position.)

   (Draw separate graphs for $t = 0$, $t = 0.5$, $t = 1$, $t = 1.5$ and $t = 2$.)

   On the same graphs, sketch the time-evolution of the stationary waves u_2 and u_3.

   How many cycles do u_2 and u_3 complete during one period of u_1? How many nodes are there? Do the number of nodes remain constant? What happens to the position of the nodes as time passes?

3. Sketch the time-evolution of the travelling wave,

$$u(x,t) = u_1(x,t) + u_2(x,t)$$

   over one period.

   Does the number of nodes remain constant? What happens to the position of the nodes as time passes?

4. For a square membrane of sidelength 1, plot $u_{1,1}$, $u_{1,2}$ and $u_{2,1}$. Print the 3-D graphs on the printer, or email them to yourself and print them on another printer.

   Are $u_{1,2}$ and $u_{2,1}$ the same, or different? If they differ, describe the difference.

   For each stationary state, view the animation. How many nodes are there? What do the nodes look like (hint: they are not points). Do they move around, or are they stationary?

5. For a square membrane, plot $u_{1,1} + u_{2,1}$ and view the animation.

   Does the number of nodes remain constant? What happens to the position of the nodes as time passes?