Reducing the Optimum: Fixed Parameter Inapproximability for CLIQUE and SETCOVER in Time Super-exponential in OPT

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Abstract

A minimization (resp., maximization) problem is called fixed parameter $\rho$-inapproximable, for a function $\rho \geq 1$, if there does not exist an algorithm that given a problem instance $I$ with optimum value $\text{OPT}$ and an integer $k$, either finds a feasible solution of value at most $\rho(k) \cdot k$ (resp., at least $k/\rho(k)$) or finds a certificate that $k < \text{OPT}$ (resp., $k > \text{OPT}$) in time $t(k) \cdot |I|^{O(1)}$ for some function $t$. In this paper, we present motivations for studying inapproximability in terms the parameter $\text{OPT}$, the optimum value of an instance. A problem is called $(r,t)$-FPT-hard in parameter $\text{OPT}$ for functions $r,t$, if it admits no $r(\text{OPT})$ approximation that runs in time $t(\text{OPT})|I|^{O(1)}$. To prove hardness, we use gap reductions from 3-SAT and assume the Exponential Time Hypothesis (ETH). It is easy to see that if the value of $\text{OPT}$ known in the ‘yes’ instance of the gap reduction, inapproximability w.r.t. $\text{OPT}$ implies the inapproximability w.r.t. input integer $k$. The converse is not true. Hence inapproximability in $\text{OPT}$ is stronger. Previous FPT-hardness results for the problems we study [5] have running times $t$ that are sub-exponential in $\text{OPT}$. Such results can often be obtained by a simple ‘translation’ of inapproximability results to FPT-hardness. In this paper, therefore, we are only interested in times $t(\text{OPT})$ that are super-exponential in $\text{OPT}$. Fellows [11] conjectured that SETCOVER and CLIQUE are $(r,t)$-FPT-hard for any pair of non-decreasing functions $r,t$ and input parameter $k$. We give the first inapproximability results for these problems with running times super-exponential in $\text{OPT}$. Since one would like to prove inapproximability with as fast growing functions $r,t$ as possible, it is critical to reduce the value of $\text{OPT}$ (relative to the gap in theOPT value between yes and no instances and the size of the instance). Our paper introduces systematic techniques to reduce the value of the optimum. These techniques are robust and work for three quite different problems. One of our results shows that, under ETH, CLIQUE is $(r,t)$-FPT-hard for $r(\text{OPT}) = 1/(1-\epsilon)$ with some constant $\epsilon > 0$ and any non-decreasing function $t$. The running time can be also set to $2^{o(n)}$, for an arbitrary $o(n)$ exponent. This improves the main result of Feige and Kilian [12] in two ways. We also show that the Minimum Maximal Independent Set (MMIS) problem is $(r,t)$-FPT-hard in OPT, for arbitrarily fast growing functions $r,t$ of OPT. While a similar result was known in terms of parameter $k$, we present this result since the reduction of $\text{OPT}$ value in this case, is very instructive and we believe will find further applications.

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1 Introduction

1.1 FPT inapproximability with parameter opt: motivation

In Fixed Parameter Tractability (FPT) theory, we are a decision problem $P$ with a parameter $k$, that relates to the problem instance. An FPT algorithm for a problem is an exact algorithm that runs in time $t(k) \cdot n^{O(1)}$ where $n$ is the size of the instance and $t$ is an arbitrary function.\(^1\) An FPT approximation algorithm, given a parameter $k$, approximates the desired solution value within a ratio $r(k)$ and runs in time $t(k) \cdot n^{O(1)}$, for a function $r$. More precisely, an FPT-approximation algorithm is defined as follows. For a minimization (resp., maximization) problem $P$, an algorithm is called an $(r,t)$-FPT-approximation algorithm for $P$ with input parameter $k$, if the algorithm takes as input an instance $I$ with (possibly unknown) optimum value $\text{opt}$ and an integer parameter $k$ and either computes a feasible solution to $I$ with value at most $k \cdot r(k)$ (resp., at least $k/r(k)$) or computes a certificate that $k < \text{opt}$ (resp., $k > \text{opt}$) in time $t(k) \cdot |I|^{O(1)}$. In the latter case, such a certificate can be obtained from the analysis of the algorithm and the fact that it did not produce the desired solution. Here the goal is to design algorithms with as slow growing functions $r$ and $t$ as possible. A problem is called $(r,t)$-FPT-inapproximable (or, $(r,t)$-FPT-hard) if it does not admit any $(r,t)$-FPT-approximation algorithm. Here the goal is to show inapproximability with as fast growing functions $r$ and $t$ as possible.

In this paper, we use reductions from 3-sat to prove FPT-hardness. When doing a reduction from 3-sat to an optimization problem $P$, a “yes” instance of $P$ is an instance obtained from a satisfiable formula, and a “no” instance of $P$ is an instance obtained from a reduction of non-satisfiable formula. Whenever we use reductions from 3-sat, we denote the number of variables by $q$, the number of clauses by $m$ and $N = m + q$.

Our work is motivated by a conjecture, by Mike Fellows, concerning parameterized approximation for setcover and clique.

**Conjecture 1.1 (FPT-hardness of setcover and clique (Fellows [11]))**

- **setcover** is $(r,t)$-FPT-hard for any non-decreasing functions $r$ and $t$.
- **clique** is $(r,t)$-FPT-hard for any non-decreasing functions $r(k) = o(k)$ and $t$.

When we do a gap reduction from 3-sat, in all cases in this paper, the optimum $\text{opt}$ of a yes instance is known. Thus, by definition, inapproximability in terms of $\text{opt}$ implies inapproximability in terms of $k$ by setting $k = \text{opt}$. On the other hand, inapproximability in terms of $k$ does not imply inapproximability in $\text{opt}$ because it may be that $k < \text{opt}$ but we can not give a certificate that $k < \text{opt}$. Typically, it happens if $k$ is only slightly smaller than $\text{opt}$. In this case, proving $\rho(k)$-FPT hardness, implies that is it is not possible to find a solution of size $k \cdot \rho(k)$. Clearly, showing that we cannot find a solution of value $\text{opt} \cdot \rho(\text{opt})$ is a stronger statement if $k < \text{opt}$. Thus if $\text{opt}$ is known inapproximability in $\text{opt}$ implies inapproximability in $k$, but not vice-versa.

Therefore, we suggest a principle of studying the hardness in terms of $\text{opt}$ whenever possible. The following version of the conjecture, is with parameter $\text{opt}$.

\(^{1}\)Unless otherwise stated, all mentioned functions are total computable functions from non-negative integers to themselves.
Conjecture 1.2 (FPT-hardness of setcover and clique with parameter OPT.) Let $\text{OPT}$ and $n$ denote the value of the optimum and size of the given instance, respectively.

- **Setcover** admits no $r(\text{OPT})$ approximation that runs in time $t(\text{OPT}) \cdot n^{O(1)}$ for any non-decreasing functions $r$ and $t$.
- **Clique** admits no $r(\text{OPT})$ approximation that runs in time $t(\text{OPT}) \cdot n^{O(1)}$ for any non-decreasing functions $r(k) = o(k)$ and $t$.

1.2 Reduction with sub-exponential time in opt and their weaknesses

Many times it is automatic to translate an inapproximability result, to FPT-hardness. This happens if we are allowed to prove $(r, t)$-FPT-hardness for time $t(\text{opt})$, sub exponential in $\text{opt}$.

Consider a polynomial time gap reduction from 3-sat to any other problem $P$. Let $|I| = q + m$ be the size of the 3-SAT instance and thus the size of the instance of $P$ is clearly bounded by $m^c$, for some constant $c$.

Many times the gap with respect to $\text{OPT}$ and the gap with respect to $n$, are about the same, as $\text{OPT}$ is close to $n$. For example, for all standard reduction from 3SAT to CLIQUE and SETCOVER, $n$ and OPT are very close. However, there are even cases in which OPT is much smaller than $n$. Thus a gap in $n$ implies a larger gap in OPT.

In any case, say that we have $\rho(\text{OPT})$ gap that resulted from a reduction from 3-sat. Thus the ETH implies that we cannot find an approximation better than $\rho(\text{OPT})$ in time $2^{o(m)}$.

Thus, all we need to do is translate $2^{o(m)}$ to a function of $\text{OPT}$. In almost all cases $\text{OPT} \leq n$ (albeit if OPT is polynomial in $n$ the next claim still holds). In that case, $\text{OPT} \leq n = m^c$. For any constant $c' > c$, $2^{\text{OPT}^{1/c'}} = 2^{o(m)}$. Thus we automatically get a $\rho(\text{OPT})$-FPT-hardness in time $2^{o(m)}$. Clearly, the meaning of such a reduction is limited and its a translation of the hardness result to FPT-hardness language. The detail that allowed us to give such a translation is that that $t(\text{OPT})$ is sub exponential in $\text{OPT}$.

1.3 Reducing the value of OPT

We are not able to prove either of these conjectures with our current techniques. It seems to us that the current state of knowledge is not enough to prove the conjectures. In fact we suspect that in order to prove this conjecture, a parameterized version of the PCP is needed. Nevertheless, we make a very important breakthrough proving hardness results with running times $t(\text{OPT})$ that are super-exponential in $\text{OPT}$. Such results were never proven before. In particular, the inapproximability of [5] is under a sub-exponential running time. We discuss this and the relation to the reducing the value of $\text{OPT}$. Proving hardness result with super-exponential time in $\text{OPT}$ or $k$ necessitates reducing the value of $\text{OPT}$. The reason is that $t(\text{OPT}) = 2^{o(m)}$ must hold with $m$ the number of clauses in the 3-SAT instance we reduce from. As the function $t$ becomes faster and faster growing, $\text{OPT}$ needs to get smaller and smaller. Therefore we claim that an important aspect of the art of proving FPT-hardness is creating new instances with smaller and smaller $\text{OPT}$.
We develop systematic techniques to reduce the value of $\text{OPT}$ for the problems we study, with the following property. Given a pair of ‘yes’ and ‘no’ instances for the problem, with optimum values $\text{OPT}_y$ and $\text{OPT}_n$ respectively, the reduction creates new instances with optimum values $\text{OPT}'_y$ and $\text{OPT}'_n$ respectively such that

- the new optimum values are much smaller: $\text{OPT}'_y \ll \text{OPT}_y$ and $\text{OPT}'_n \ll \text{OPT}_n$,

- the gap between optimum values in yes and no instances is preserved: $\text{OPT}'_y / \text{OPT}'_n \approx \text{OPT}_y / \text{OPT}_n$,

- the new instances sizes are not much larger than the old ones.

No FPT hardness before us used any techniques to reduce $\text{OPT}$, even though reducing the $\text{OPT}$ seems to us to be one of the most natural and important ideas for proving FPT-hardness.

Our approach can be summarized as follows.

1. Try to prove inapproximability in terms of $\text{OPT}$ and not in terms of $k$. For this you need gap reductions in which the value of a “yes” instance is known (it is also possible to do reductions in terms of $\text{OPT}$ if its only approximately known but its much better if $\text{OPT}$ is known).

2. Prove only inapproximability in time super-exponential in $\text{OPT}$ or larger.

3. An important tool in proving a strong FPT-hardness is developing techniques to reduce an instance of a problem to another instance of the same problem, with much smaller optimum, while not losing much in the gap.

Apart from SETCOVER and CLIQUE, we also study a problem called the Minimum Maximal Independent Set (MMIS) problem. In this problem, given an undirected graph, the goal is to find a minimum-size independent set that is also inclusion-wise maximal. This is also equivalent to finding the minimum-size independent set that is also a dominating set.

## 2 Previous work

The following relation is known among the parameterized complexity classes: $\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2]$. It is widely believed that $\text{FPT} \neq \text{W}[1]$. In fact $\text{FPT}=\text{W}[1]$ implies that $\text{ETH}$ fails.

**Inapproximability that is sub-exponential in $n$:** A widely explored line of research shows, for CLIQUE and SETCOVER, a relation between an approximation, and the running time it requires. Such results are discussed in [10]. Recently, [2] improved [10] to get the following result. For any $r$ larger than some constant and any constant $\epsilon > 0$, any $r$-approximation algorithm for the maximum independent set problem must run in time at least $2^{n^{1-\epsilon}/r^{1+\epsilon}}$. This nearly matches the upper bound of $2^{n/r}$. In this case super exponential running times are out of the question, because the time depends on $n$. This again shows the power of parameterizing algorithms. In the instance we start with, the optimum is very close to $n$. By reducing $\text{OPT}$ we can get inapproximability for CLIQUE and SETCOVER in time super-exponential in $\text{OPT}$, giving a more refined classification of the problems.

To the best of our knowledge, the effort of showing FPT-hardness for CLIQUE and SETCOVER (in terms of $k$ and $\text{OPT}$) started with [5].
Theorem 2.1 [5] Under ETH and PGC, there exist constants $1 > F_1, F_2 > 0$ such that the setcover problem does not admit an FPT approximation algorithm with ratio $OPT^{F_1}$ in time $2^{OPT^{F_2}}$.

The above theorem uses $F_2 < 1$ hence uses time sub-exponential time $OPT$ and is not suited for this paper.

Theorem 2.2 [5] Unless $NP \subseteq \text{SUBEXP}$, for every $0 < \delta < 1$ there exists a constant $F = F(\delta) > 0$ such that clique admits no FPT approximation within $OPT^{1-\delta}$ in time $2^{OPT^F}$.

As $F < 1$ in the above construction the running time here too is sub-exponential in $OPT$ and this theorem is not suited for this paper.

These theorems seem unrelated to the results of [10] and [2] because of the large inapproximability proved in the theorems, compared to the constant lower bound in the above papers.

Previous work on MMIS: The problem is known to be $W[2]$-Hard [13]. It is also known that it is $(r(k), t(k))$-FPT-hard for any increasing $r, t$. The $r(k)$ inapproximability for any $r$ was shown in [9], under the assumption that $\text{FPT} \neq W[2]$. Indeed, reducing directly from $W[1]$ or $W[2]$ creating a gap, is another important technique to get FPT-hardness (if somewhat hard to use). The arbitrary time $t(k)$ follows from proving that the ETH implies $\text{FPT} \neq W[2]$. This proof is covered by the papers [4, 3, 8] and the Ph.D. thesis of Lokshatanov [10]. The version we discuss is from the survey paper http://www.cs.bme.hu/ dmarx/papers/survey-eth-beatcs.pdf.

The proof in that survey first shows that if Clique admits an FPT algorithm, then 3-coloring admits a $2^{o(m)}$ time algorithm which implies that 3-SAT admits a $2^{o(m)}$ exact solution. This is due to the simple linear reduction from 3-SAT to 3-Coloring. This in turn implies that the ETH fails. Thus assuming ETH, Clique admits no FPT algorithm. Then it is shown that if Clique has no FPT algorithm a problem called Multicolored Clique admits no FPT algorithm. Finally it is proved that the if the Multicolored Clique problem admits no FPT algorithm, the Dominating Set (hence setcover as well) admits no FPT algorithm.

We prove a very slightly stronger statement namely $(r(OPT), t(OPT))$-FPT-hard for any increasing $r, s$ with the optimum known. However, this slight improvement is not the reason we include that proof. The reduction of the optimum in this case is quite non-trivial and requires ideas that probably will find future applications. To understand the difficulty consider a reduction from 3-SAT to MMIS in which we basically try to transform the instance into one in which literals (that will be the minimal independent set) will dominate clauses that they belong to. A great difficulty is that we need to choose a small number of the literals, only. The literals not chosen must be dominated as well. Thus, we need to add edges between literals. But then: how do we make sure that the independent set has no edges between literals in the set, and on top of that make the optimum much smaller?

While this difficulties can be overcome, its far from easy. From a technical viewpoint, the proof that in MMIS we can reduce OPT to be smaller than any $f(n)$ that we wish, is the most challenging proof in the paper.

In [5] a large collection of $W[1]$-hard problems are presented for which an inapproximability such as in the Fellows conjecture does not apply. In fact, these problem are given some approximation $f(OPT)$ for $f(x) \leq x^2$ and the running time is just polynomial in the size of the input. All these problems are not only $W[1]$-hard, but also admit strong inapproximability results (at least Label-Cover hardness or believed to have no better than polynomial ratio like the minimization version of the Dense $k$-subgraph problem).
While all problems of [5] are minimization problems, such results hold for maximization problems as well, as the next example shows. For a set \( U \), the edges \( e(U) \) are the edges with both endpoints in \( U \). The parameterized version of the Dense \( q \)-subgraph is defined as follows. The input is a simple connected graph \( G(V,E) \) and parameters \( q \) and \( p \). The question is whether there is a set \( U \) with \( q \) vertices so that, \( e(U) \) is at least \( p \)? Clearly this problem is \( W[1] \)-hard as \( \text{CLIQUE} \) is a special case of it. But returning a spanning tree on any \( q \) vertices gives \( \text{OPT} + 3 \) approximation. To see that note that the number of edges in such a tree is \( q - 1 \). In addition, \( p < q^2 \) as if \( p > q^2 \) we can say no immediately as no set with \( q \) vertices and \( q^2 \) edges exits. In other words, \( q - 1 \leq \text{OPT} \leq q^2 \) for the instance. Since the number of edges returned is \( q - 1 \), the ratio is \( p/(q - 1) \leq q^2/(q - 1) \leq q + 2 \). This implies a ratio of \( \text{OPT} + 3 \) as \( \text{OPT} \geq q - 1 \). The time is polynomial in \( n = |V| \). For future reference, the minimization version of the Dense \( k \)-subgraph problem admits as an input a connected graph and two parameters \( Q \) and \( p \). The question is if there exists a set \( U \) of size \( p \), so that \( e(U) > Q \).

**Theorem 2.3** [5] Dense \( k \)-subgraph, Directed Multicut, Directed Steiner Tree, Directed Steiner Forest, Directed Steiner Network and the minimization version of the Dense \( k \)-subgraph problem, admit \( g(\text{OPT}) \)-approximation algorithms that runs in polynomial time, for some small function \( g \) (the largest approximation ratio we give is \( \text{OPT}^2 \)).

**Approximation under FPT time:** The Strongly connected Steiner Subgraph problem is given a graph \( G(V,U) \) with a set of terminals \( T \subseteq V \), find a strongly connected subgraph that includes all the terminals. It is elementary to see that this problem is equivalent with respect to approximation to the Directed Steiner Tree problem that admits no better than \( \log^{2-\epsilon} n \) ratio for any constant \( \epsilon \) [15].

**Theorem 2.4** [5] Strongly Connected Steiner Subgraph problem is \( W[1] \)-hard and does not admits a better approximation than \( \log^{2-\epsilon} n \) for any constant \( \epsilon \). However, if FPT time is allowed we can get a 2-approximation algorithm with running time \( h(\text{OPT}) n^{O(1)} \).

So far this seems to be the only natural problem for which such a result is known where allowing FPT times reduces the order of magnitude the approximation factor. This gives evidence that we should try to find FPT approximations for \( W[1] \) or \( W[2] \) hard problems that also admit a strong inapproximability. Maybe in FPT time, a better approximation is possible?

**Previous work from inapproximability theory:**

**Notation:** Throughout this paper, the number of sets in a SETCOVER instance and the number of vertices in a CLIQUE instance will be denoted by \( n \).

**Theorem 2.5** [16, 24] Unless \( \text{P} = \text{NP} \), CLIQUE can not be approximated within \( n^{1-\epsilon} \).

The reduction in [16] is randomized but in [24] this result is derandomized and it is achieved under the assumption that \( \text{P} \neq \text{NP} \). Note that there is a stronger inapproximability result for CLIQUE, e.g., in [18], but we can not use it because the running time of the reduction is quasi polynomial (in fact if the reduction would have been polynomial, this would contradict ETH).

**Theorem 2.6** [23] Unless \( \text{P} = \text{NP} \), SETCOVER admits no better than \( c \log n \) approximation for some constant \( c \).

We note that the inapproximability results of Theorems 2.5 and 2.6 are close to the best possible algorithmic results. CLIQUE admits a trivial \( n \) ratio approximation. For SETCOVER, it is known (folklore) that the natural greedy algorithm gives a \( \ln n + 1 \) approximation.


3 Preliminaries

We begin by describing the complexity theoretic conjectures assumed in proving the hardness results in this paper. Impagliazzo et al. [17] formulated the following conjecture which is known as ETH. We assume ETH in all hardness results in this paper.

**Exponential Time Hypothesis (ETH)**

$3$-SAT cannot be solved in $2^{o(q)(q + m)}O(1)$ time where $q$ is the number of variables and $m$ is the number of clauses.

Using the Sparsification Lemma of Calabro et al. [1], the following lemma follows.

**Lemma 3.1** Assuming ETH, $3$-SAT cannot be solved in $2^{o(m)(q + m)}O(1)$ time where $q$ is the number of variables and $m$ is the number of clauses.

**Remark on the relation between $N$ and $m$:** We may assume without loss of generality that there are more clauses than variables in the SAT instance. The other case is similar. Thus if the number of variables is $q$ we get $N = q + m \leq 2m$. Hence, we do not need to use $N$, and $m$ can replace it in any future computation (the factor 2 difference is never significant in this paper).

Another conjecture used is Projection Game Conjecture (PGC) due to Moshkovitz [21].

**Projection Game Conjecture (PGC):**

There exists some constant $c$ and PCP of size $m \cdot \rho(m) \cdot \text{poly}(1/\varepsilon)$, with soundness $1/\varepsilon$ for any $\varepsilon$ so that $1/\varepsilon \leq m^c$. The alphabet is of size $\text{poly}(1/\varepsilon)$ and $\rho(m) = 2^{\log^\alpha m}$, for a constant $\alpha$, $0 < \alpha < 1$. The PCP is equivalent to the Labelcover problem with the projection property.

In [21] the sublinear term $\rho(m)$ was not discussed and thus we took the sublinear term from [22]. The conjecture is actually already proven in [22] if we allow alphabet exponential in $1/\varepsilon$.

The MIN-REP problem was first defined in [19] where it is shown that if Labelcover (see [19]) has gap $1/\varepsilon$, MIN-REP has gap $\sqrt{1/\varepsilon}$. The MIN-REP problem is a problem in graphs and is easy to understand, which is the reason it is defined as such in [19].

**Min-Rep (MIN-REP)**

The instance is a bipartite graph $G = (U, W, E)$ where $U$ and $W$ are each split into a disjoint union of $q$ sets $U = \cup_{i=1}^q A_i$ and $W = \cup_{i=1}^q B_i$. The sets $A_i$ and $B_i$ are called supervertices. The graph $G$ and the partition into supervertices induce edges $E^+$ on supervertices: for two supervertices $A_i$ and $B_j$, we have superedge $(A_i, B_j) \in E^+$ if there exist $a \in A_i$ and $b \in B_j$ such that $(a, b) \in E$. We say that a subset $S \subseteq U \cup W$ covers a superedge $(A_i, B_j) \in E^+$ if there exist $a \in A_i \cap S$ and $b \in B_j \cap S$ so that $(a, b) \in E$. The goal in MIN-REP is to find a minimum-size set $S \subseteq U \cup W$ that covers all superedges in $E^+$.

The result of [19, 21], imply the following reduction from PGC to MIN-REP. The PCP in question can be posed as a Min-Rep instance because the PCP is queried in two places. The number of supervertices in the supergraph equals the size of the PCP. Thus by [21], the number of questions in the MIN-REP instance is $O(m \cdot \text{exp} (\log^\alpha m) \cdot \text{poly} \log(m) \cdot (1/\varepsilon)^{cn})$ for some constant $c$. 


Because we deal with the PGC, we may assume that the number of vertices that belong to any supervertex of the MIN-REP graph is \((1/\epsilon)^{c_2}\) for some constant \(c_2\). Thus the size of the MIN-REP graph, namely, the number of vertices is \(O(m \cdot \exp(\log^a m) \cdot \log(m) \cdot (1/\epsilon)^{c_1+c_2})\).

In a “yes” instance, it is possible to choose one vertex per supervertex and cover all the superedges, thus the optimum is \(O(m \cdot \exp(\log^a m) \cdot \log(m) \cdot 1/\epsilon^{c_1})\).

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<th>Reduction from 3-SAT to MIN-REP [19, 21]</th>
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| Given an instance \(I\) of 3-SAT with size \(m\) equal to the number of clauses in \(I\), for constants \(c_1, c_2, c_3, \alpha > 0\) and any choice of \(\epsilon\) so that \(\epsilon > 1/m^{c_3}\), there exists a “yes” instance of MIN-REP which admits a feasible solution of size \(\sigma = O(m \cdot \exp(\log^a m) \cdot \log(m) \cdot 1/\epsilon^{c_1})\). The optimum for a “no” instance of MIN-REP is at least \(\Omega(1/\sqrt{\epsilon})\) times the value of a “yes” instance. The projection property is translated here to the following. The graph induced by any \(A_i \cup B_j\) that are a super edge is a union of disjoint stars with heads in \(A_i\). By composing the above reduction with a reduction to MIN-REP of [19], with the reduction from MIN-REP to SETCOVER [20], the following corollary is derived:

**Corollary 3.2** There exists a reduction from 3-SAT to SETCOVER with \(m\) clauses so that

1. The number of sets is \(m \cdot 2^{\log^a m} \cdot \log(m) \cdot (1/\epsilon)^{c_1+c_2}\) with \(c_1\) and \(c_2\) some constants larger than 0 and \(\alpha\) a constant that satisfies \(0 < \alpha < 1\).
2. The number of elements is \(\text{poly}(m)\).
3. The optimum for a “yes” instance is exactly \(m \cdot 2^{\log^a m} \cdot \log(m)(1/\epsilon)^{c_1}\).
4. The optimum for a “no” instance is at least \(d \cdot \sqrt{\epsilon} \cdot m \cdot 2^{\log^a m} \cdot \log(m) \cdot (1/\epsilon)^{c_1}\) for a constant \(d > 0\).

As we shall later see, we choose \(\epsilon = c^2/\log^2 m\) for a constant \(c > 0\). The inapproximability of SETCOVER becomes \(c \cdot \log m\). In addition, since \(1/\epsilon\) is by itself polylogarithmic in \(m\), we can denote the size of the optimum for a “yes” instance by \(m \cdot 2^{\log^a m} \cdot \log(m)\).

We briefly explain how does the [20] reduction works.

Every vertex in the MIN-REP instance is a set in the reduction of [20]. As the size of the MIN-REP graph is \(m \cdot 2^{\log^a m} \cdot \log(m)(1/\epsilon)^{c_1+c_2} = m \cdot 2^{\log^a m} \cdot \log(m)\), this is also the number of sets in the SETCOVER instance. For every superedge \(A_i, B_j\) we add a set \(M_{i,j}\) of elements. Note that the number of superedges is no larger than the number of supervertices pairs, and so is no larger than \(m^3\). The size of every \(M_{i,j}\) is \((1/\epsilon)^{c_2} = \text{poly}(m)\) and the total number of elements is \(m^3 \cdot \text{poly}(m) = \text{poly}(m)\).

In [20] every \(a_i \in A_i, b_j \in B_j\) so that \((a_i, b_j) \in E\) are connected to two disjoint random halves of \(M_{i,j}\) (the reduction we described is randomized for simplicity of the explanation. However, this construction can be derandomized). For a “yes” instance we can choose a single vertex out of every supervertex and cover all superedges (see both [20] and [21]). Thus for a “yes” instance we can pick a SETCOVER of size \(m \cdot 2^{\log^a m} \cdot \log(m)\), because all superedges are covered, and thus there will be \(a_i, b_j\) so that \(a_i\) covers some random half and \(b_j\) covers the other half.

The \(\sqrt{1/\epsilon}\) gap for MIN-REP implies that in a “no” instance, unless you take \(\Omega(\log m)\) times the optimum value for a “yes” instance sets, you hardly cover any of the superedges. We choose
$|M_{ij}| = 2^{c \log m}$ for some constant $c$. In the “no” instance, since most superedges will not be covered (there will not be $a_i, b_j$ in the solution so that $b_j$ belongs to the star of $a_i$) $M_{ij}$ has to be covered by a collection of random independent sets of size $|M_{ij}|/2 = 2^{c \log m}/2$. Using random halves, about $c \cdot \log m$ sets are required to cover a single $M_{ij}$ which is the source of the gap.

4 Our results

Recall that we call an optimization problem $(r, t)$-FPT-hard if it admits no algorithm with approximation ratio $r(\text{OPT})$ and running time $t(\text{OPT}) \cdot n^{O(1)}$, where $\text{OPT}$ is the optimum of some instance, $n$ is the size of the given instance and $r, t$ are given functions. In all our reductions $\text{OPT}$ is known and so the reduction implies a similar hardness for a parameter $k$.

Theorem 4.1 Under ETH and PGC, setcover is $(r, t)$-FPT-hard for $r(\text{OPT}) = (\log \text{OPT})^\gamma$ and $t(\text{OPT}) = \exp(\exp((\log \text{OPT})^\gamma)) = \exp\left(\text{OPT}^{(\log f(\text{OPT}))}\right)$ for some constant $\gamma > 1$ and $f = \gamma - 1$.

The time here is much larger than just exponential in $\text{OPT}$.

Theorem 4.2 Under ETH and a stronger version of PGC with PCP length $O(m \cdot \text{poly log}(m) \cdot \log(1/\epsilon))$ and gap $\Omega(1/\epsilon)$ for any $\epsilon \geq 1/m^c$, for some constant $c$, setcover is $(r, t)$-FPT-hard for $r(\text{OPT}) = \text{OPT}^d$ and $t(\text{OPT}) = \exp(\exp(\text{OPT}^{d''}))$ for some constants $d', d'' > 0$.

This kind of PCP was conjectured to exist by Moshkovitz in a private communication.

Note that the running times in this result is almost doubly exponential in $\text{OPT}$. We later show this result is basically the best we can expect if we just use PCP (even under the best conceivable PCP).

We can also prove an inapproximability with super-exponential time in $\text{OPT}$ that only assumes ETH.

Theorem 4.3 Under ETH alone, setcover cannot be approximated within

$$c \sqrt{\log \text{OPT}}$$

for some constant $c$, in time $\exp\left(\text{OPT}^{(\log \text{OPT})^f}\right)$ for $f$ the same constant from Theorem 4.1.

Here the inapproximability we get is significantly smaller if we can not assume the PGC. But the running time is the same, hence super-exponential.

Theorem 4.4 Under ETH, clique is $(r, t)$-FPT-hard for $r(\text{OPT}) = 1/(1-\epsilon)$ for some constant $\epsilon$, that satisfies $0 < \epsilon < 1$, and any non-decreasing function $t$, however huge. The running time can also be set to $2^o(n)$ of our choice of $o(n)$.

It is interesting to compare this result to the paper by Feige et al [12]. In [12] it is shown that for $\text{OPT} \leq \log n$, CLIQUE can not be solved exactly in time significantly smaller than $n^{\text{OPT}} < n^{\log n}$ time, unless $\text{NP}$ has sub-exponential simulation. In fact the statement is much stronger than that. If CLIQUE can be solved in much smaller time than $n^{\text{OPT}}$, any solution for an NP problem that makes $f(n)$ non-deterministic time, implies a deterministic solution in time roughly $\exp(\sqrt{f(n)})$. This implies a host of NPC problems admits a sub-exponential algorithm including 3-SAT (the number of non deterministic bits used in 3-SAT is clearly at
most $n$). Therefore it is possible to prove [12] under ETH (which is our standard assumption in this paper) as the contradiction in [12] implies that ETH is not valid.

Theorem 4.4 works for any OPT and OPT $\leq \log n$ in particular, and thus improves the paper of Feige et al [12] in two ways. First we prove $1/(1 - \epsilon)$-hardness which for such small values of OPT might be significantly harder than ruling out an exact solution. Second, the $r(OPT)$-hardness holds even if we allow time $2^\text{opt}$ time. This is a time much much larger than $n^{\log n}$ of [12].

Clarification: It may seem strange that given time $2^\text{opt}$, it is still not enough to exhaustively search for an optimum clique. However, as a first step of our algorithm a graph of size $2^\text{opt}$ is constructed. Searching for the optimum exhaustively in such a graph requires exponential in $n$ time and does not contradict ETH.

Theorem 4.5 Let $r(OPT) = \left(\frac{1}{1 - \epsilon}\right)^{\log^{1/3} \text{OPT}}$, with $\epsilon$ the constant from Theorem 4.4. Then CLIQUE is $(r,t)$-FPT-hard. for any function $t$, however huge.

As a function of $n$, we later show that the time can be set to $2^{n^{1/Q(n)}}$ for an arbitrarily slowly growing $Q(n)$. Thus Theorem 4.5 improves [12] in the same two ways that we mentioned for Theorem 4.4. The inapproximability is now super constant, versus an exact solution, and the running time is still much much higher than $n^{\log n}$.

We study a well known $W[2]$ hard problem mmis, and give hardness in terms of OPT.

Theorem 4.6 Under ETH, mmis is $(r,t)$-FPT-hard in OPT (and thus in $k$ since OPT is known) for any non-decreasing functions $r$ and $t$.

It is our opinion that one must try to prove as many such results to as possible. This may shade light on how to prove this for CLIQUE and SETCOVER.

Avoiding a constant optimum: Another standard we think is good imposing in case of FPT-hardness is that the optimum for a "yes" instance is not constant. To explain that, we consider the 3-Coloring problem and (trivially) show that Fellows conjecture holds for this problem. In [7] it is shown that 3-coloring admits no constant approximation for any constant unless a variant of the (well known) Khot two-to-one PGC holds. Take the "yes" instance of the problem (in which OPT = 3). For any $r$, $r(OPT) = g(3)$ is a constant and for any $t$, $t(OPT) \cdot n^{O(1)}$ is just polynomial time. By the above hardness of [7], it is clear that for any $r$, $t$ 3-coloring is $(r,t)$-FPT-hard. The lesson to be derived from this example is that we should only try hardness for problems with non-constant OPT.

4.1 Reducing the value of OPT while proving FPT-hardness

Our proofs are based on gap-reductions from 3-SAT to the given optimization problem. To describe our technique, we define a notion of gap reductions as follows. Fix a minimization problem $P$ and two non-decreasing functions $r$ and $t$. We use $m$ for the number of clauses in the 3-SAT instance we reduce from.

Definition 4.7 (Gap reduction) Let $m$ be the number of clauses in the 3-SAT problem we reduce from. An algorithm that maps any given instance $I$ of 3-SAT to an instance $M_I$ of problem
If there exists a gap-reduction (according to Definition 4.7) for a minimization problem \( P \) with non-decreasing functions \( \kappa_T \) and \( \kappa_F \) such that:

- \( t(\kappa_T(m)) = 2^{o(m)} \) and \( \kappa_T(m) \cdot r(\kappa_T(m)) < \kappa_F(m) \),
- The algorithm takes \( 2^{o(m)} \) time to construct \( M_I \) (and hence the size of \( M_I \) is \( 2^{o(m)} \)),
- \( \text{OPT}(M_I) \leq \kappa_T(m) \) if \( I \) is satisfiable,
- \( \text{OPT}(M_I) \geq \kappa_F(m) \) if \( I \) is unsatisfiable.

We now prove the following simple but useful claim.

**Claim 4.8** If there exists a gap-reduction (according to Definition 4.7) for a minimization problem \( P \) with non-decreasing functions \( r \) and \( t \), then assuming ETH, problem \( P \) is \((r,t)\)-FPT-hard with parameter \( \text{OPT} \).

**Proof:** Assume on the contrary that there is an \( r(\text{OPT}) \)-approximation algorithm \( A \) with running time \( t(\text{OPT}) \cdot m^c \) for \( P \) where \( \text{OPT} \) is the value of the optimum, \( m \) is the size of the given instance of \( P \) and \( c > 0 \) is a constant. Now we design an algorithm for 3-SAT in time \( 2^{o(m)} \) where \( m \) is the number of clauses, as follows. Given a 3-SAT instance \( I \), we first use the gap-reduction to compute instance \( M_I \) of \( P \) and then run algorithm \( A \) on \( M_I \) for time \( t(\kappa_T(m)) \cdot m^c \) where \( m \) is the size of \( M_I \). Since \( A \) is an \( r(\text{OPT}) \)-approximation, we can decide whether \( \text{OPT}(M_I) \leq \kappa_T(m) \cdot r(\kappa_T(m)) < \kappa_F(m) \). Thus we can decide whether \( I \) is satisfiable. Note that \( A \) takes time \( 2^{o(m)} \), contradicting ETH. This finishes the proof. \( \square \)

## 5 Inapproximability for Set Cover with super-exponential time in OPT

In this section we prove Theorem 4.1.

Corollary 3.2 implies the following corollary.

**Corollary 5.1** There exists constants \( c_1, c_2 > 0 \) and a constant \( 0 < \alpha < 1 \), so that the following holds. Let \( m \) be the number of clauses in the 3-SAT problem we reduce from. Assuming PGC and ETH, there exists a reduction from 3-SAT to SETCOVER so that the number of sets in the resulting instance is \( \sigma = m \cdot 2^{\log^\alpha m} \cdot \text{poly log}(m) \cdot (1/\epsilon)^{c_1 + c_2} \) for two constant \( c_1, c_2 > 1 \). Furthermore, value of the optimum in “yes” instance is exactly \( \kappa = m \cdot 2^{\log^\alpha m} \cdot \text{poly log}(m)(1/\epsilon)^{c_1} \) and that in the “no” instance is at least \( c \cdot \log m \cdot \kappa \).

We use \( \epsilon = c^2 / \log^2 m \) here.

We now describe a way to change the SETCOVER instance so that we can use Claim 4.8. The idea is to make the optimum much smaller. Starting with the SETCOVER instance \( S = (U, S) \) in the above corollary, where \( U \) is the set of elements and \( S \subseteq 2^U \) is the collection of sets, we construct a new instance \( S' = (U, S') \) on the same elements as follows. We introduce a set \( s \in S' \) as \( s = \cup_{i=1}^{p} s_i \) for each subcollection \( \{s_1, s_2, \ldots, s_p\} \subseteq S \) of size \( p \) where \( 1 \leq p \leq \lfloor m / \log m \rfloor \).

**Claim 5.2** Number of sets in the new instance \( S' = (U, S') \) is \( 2^{o(m)} \). The new instance can be constructed in time \( 2^{o(m)} \).
Proof: Recall that the number of sets in the original instance is \( \sigma = m \cdot 2^{\log^* m} \cdot \text{poly}(\log(m)) \) because of the choice of \( \epsilon \). Thus since \( p \leq \sigma / 2 \), the number of sets in the new instance is

\[
\sum_{p=1}^{\lfloor m/\log m \rfloor} \binom{\sigma}{p} \leq \lfloor m/\log m \rfloor \cdot \left( m \cdot 2^{\log^* m} \cdot \text{poly}(\log(m)) \right)^{\lfloor m/\log m \rfloor}.
\]

We use the inequality \( \binom{n}{p} \leq (ne/p)^p \) to upper-bound this by

\[
\lfloor m/\log m \rfloor \cdot \left( e \cdot 2^{\log^* m} \cdot \text{poly}(\log(m)) \cdot 2 \log m \right)^{m/\log m} = 2^{O(m/\log^* m)} = 2^{o(m)},
\]

as claimed, where we have \( 2 \log m \) in the first expression (instead of just \( \log m \)) because of the floor function and the last equality holds since \( 0 < \alpha < 1 \). It is easy to see that the new instance can be created in \( 2^{o(m)} \) time.

\[\square\]

Claim 5.3 Proof of Theorem 4.1. The problem is \((r, t)\)-FPT-hard for \( r(k) = (\log k)^\gamma \) and \( t(k) = \exp(\exp((\log k)^\gamma)) \) for any \( 1 < \gamma < 1/\alpha \).

Proof: Clearly, any optimum will use as few sets of size (roughly) \( m/\log m \) and so the gap between a “Yes” instance and a “No” hardly changed. Namely, \( \text{OPT} \) and that of the new instance \( \text{OPT}_2 \) are related as \( \text{OPT}_1/\lfloor m/\log m \rfloor \leq \text{OPT}_2 \leq \lfloor \text{OPT}_1/\lfloor m/\log m \rfloor \rfloor \). Therefore the gap between the new optima of a “yes” instance and a “no” instance continues to be \( c' \log m \) for some constant \( c' > 0 \) and the new optimum of the “yes” instance is at most \( \kappa_T = \lfloor \kappa/\lfloor m/\log m \rfloor \rfloor = O(\log^* m \cdot \text{poly}(\log(m))). \) and \( \kappa_N \) is \( c' \cdot \log m \) larger than that.

Now define two functions \( r(k) = (\log k)^\gamma \) and \( t(k) = \exp(\exp((\log k)^\gamma)) \) for any \( 1 < \gamma < 1/\alpha \), as given in Theorem 4.1. Note that \( r(\kappa) = O((\log^* m)^\gamma) = o((\log^* m)^{1/\alpha}) = o(\log m) \) and \( t(\kappa) = 2^{o(m)} \). Thus this reduction satisfies all the conditions in Definition 4.7 for Claim 4.8 to hold. Thus \text{setcover} is \((r, t)\)-FPT-hard for these functions, proving Theorem 4.1. \[\square\]

5.1 Proof of Theorem 4.2

For proving this theorem we assume:

Conjecture 5.4 There exists a constant \( c > 0 \) and a PCP of size \( m \cdot \text{poly}(\log(m))\text{poly}(1/\epsilon) \), for any \( \epsilon \) so that \( \epsilon \geq 1/m^c \).

Is the conjecture reliable?

This result was conjectured to hold by Moshkovitz in a private communication. Note that there exists already a PCP of size even smaller than the above. In fact in [6] a PCP is presented whose size is is \( m \cdot \text{poly}(\log(m)) \). The down size is that the inapproximability of this PCP [6] is 2. Improving the inapproximability to polylogarithmic does not seem far fetched.

We now use the above conjecture and show a much stronger FPT inapproximability for \text{setcover}. By Corollary 3.2, and the above conjecture we get the following corollary, using \( \epsilon = c^2/\log^2 m \):

Corollary 5.5 There exists a constants \( c, c_1 > 0 \) and a reduction from 3-SAT to \text{setcover} so that:

1. The number of sets is \( \sigma = m \cdot \text{poly}(\log(m)) \cdot (1/\epsilon)^{c_1+c_2} = m \cdot \text{poly}(\log(m)) \) for some constants \( c_1, c_2 \).
2. The number of elements is \( \text{poly}(m) \).

3. The value of the optimum in “yes” instance is exactly \( \kappa_Y = m \cdot \text{poly log}(m) \) and that in the “no” instance is at least \( c \cdot \log m \cdot \kappa \) with \( c \) some constant \( c > 0 \).

**Proof of Theorem 4.2:** Make every collection of sets of size \( m/(d \cdot \log \log m) \) one big ‘collection set’, with \( d \) a large enough constant. Here we omit the floor and the ceiling as, in the previous proof, we saw that they hardly make a difference, and the correction needed is minimal.

The number of sets in the instance is:

\[
\left( \frac{m \cdot \text{poly log}(m)}{d \cdot \log \log m} \right)
\]

and is \( 2^{o(m)} \) if \( d \) is large enough. This is implied by the inequality \( \binom{n}{k} \leq (ne/k)^k \). The reason for the major improvement is that the term \( 2^{\log^a m} \) is gone.

After this change, the size of the optimum for a “yes” instance becomes \( \text{poly log}(m) \). Recall that the gap is \( c \log m \). Therefore, the gap can be stated as \( \text{OPT}^{d'} \) for some \( d' < 1 \).

Let \( d'' \) be any constant \( d'' < d' \). As for the running time, we use \( \text{OPT}^{d''} = c \log m \), we get \( 2^{\text{OPT}^{d''}} = o(m) \) and \( \exp(2^{\text{OPT}^{d''}}) = 2^{o(m)} \). This ends the proof of Theorem 4.2.

5.2 An inapproximability under the Exponential Time Hypothesis only

For the (maybe unlikely) case that PGC will be proved wrong, we now prove a somewhat weaker inapproximability for SETCOVER assuming ETH only. This result will remain valid even if PGC is disproved.

The following is proved in [22].

**Theorem 5.6** There exists a constant \( c \) and a PCP of size \( m \cdot 2^{\log^a m} \cdot \text{poly log}(m) \text{poly}(1/\epsilon) \), such that the size of the alphabet is at most \( \exp(1/\epsilon) \) and the gap that can be chosen to be \( 1/\epsilon \) for any \( \epsilon > 1/m^c \).

The difficulty now is that choosing too large \( \epsilon \) increases the number of sets very much. Indeed, the number of sets equals the number of vertices in MIN-REP and this number is now

\[
m \cdot 2^{\log^a m} \cdot \text{poly log}(m) \text{poly}(1/\epsilon) \exp(1/\epsilon).
\]

We choose \( \epsilon = \ln 2 \cdot \log^a m \). Then using a reduction from 3-SAT to SETCOVER described in Corollary 3.2 we get:

**Corollary 5.7** There exists a constant \( d > 0 \), and a constant \( 0 < \alpha < 1 \) and a reduction from 3-SAT to SETCOVER so that:

1. The number of sets is \( m \cdot 2^{2\log^a m} \cdot \text{poly log}(m) \).

2. The number of elements is \( \text{poly}(m) \).
3. The gap is $d \cdot \sqrt{\log \alpha m}$.

4. The optimum of a “yes” instance does not change, namely, $\text{opt} = 2^{\log^\alpha m} \cdot \text{poly log}(m)$.

The proofs here are simple computations using the new value of $\epsilon$ plugged in Corollary 3.2. The optimum $\text{opt}$ does not change because it does not depend on the alphabet. The reason is, that any optimal solution still takes one vertex from any supervertex hence the optimum for a “yes” instance is still the number of super vertices.

The inapproximability in terms of $\text{opt}$: The gap is $d \sqrt{\log \alpha m}$ for some constant $d$. $\text{opt} = 2^{\log^\alpha m} \text{poly log}(m)$. Thus for some constant $c$, the problem is $c \cdot \sqrt{\log \text{OPT}}$-hard.

The time in terms of $\text{opt}$: Since $\text{opt}$ did not change we derive exactly the same time as in Theorem 4.1, namely, $\exp(\text{opt}(\log \text{opt}))$ for the same constant $f > 0$, that appears in Theorem 4.1.

This proves Theorem 4.3.

5.3 Discussion of this specific technique

This technique alone, combined with the type of PCP used cannot be used to prove Fellows conjecture for SETCOVER because of the limitation of the PCP. Only a linear reduction from 3-SAT to SETCOVER can be used to prove the conjecture. However it is known (folklore) that such reduction can not exist as it contradicts ETH.

There is a very strong evidence that a linear PCP can not exist. The ultimate PCP we may expect (albeit this is not known even for constant $\epsilon$) is a PCP of size $m \cdot \text{poly}(1/\epsilon)$ with gap $1/\epsilon$. For the choice of $1/\epsilon = \text{poly log}(m)$ the inapproximability is almost the same as in Theorem 4.2. The running time too is doubly exponential. This shows that Theorem 4.2 got almost the best result possible if we only use almost linear PCP and our technique. It may be that the current state of PCP theory does not allow the proof of Fellows conjecture for CLIQUE or SETCOVER. A new type of PCP, namely, a parameterized version of the PCP theorem may be required.

6 A constant lower bound for Clique in arbitrarily large time in opt

We use the basic PCP theorem:

**Theorem 6.1 (The standard pcp theorem)** There exists a reduction from any NP-complete language $L$ to 3-SAT so that a “yes” instance is mapped to a 3-SAT instance such that all clauses can be simultaneously satisfied, while a “no”-instance is mapped to an instance such that at most $1 - \epsilon$ fraction of the clauses can be simultaneously satisfied. Here $\epsilon > 0$ is a constant.

We now describe (for the sake of completeness) a totally standard reduction from 3-SAT to CLIQUE. In the reduction, for each clause $C$ in the 3-SAT instance, we add a set of seven new vertices $V_C$ – one vertex for each of the seven satisfying assignments to the three variables in the clause. Thus each vertex corresponds to a partial assignment, i.e., truth-assignment to three variables. We add an edge between two vertices if the corresponding partial assignments are ‘compatible’, i.e., they have a common extension as a full assignment. Note that if two
clauses $C_1$ and $C_2$ do not share any variables, there is a complete bipartite graph between the corresponding sets of seven vertices. Note also that $V_C$ forms an independent set for any clause $C$, because they, by definition, must disagree on the value of at least one variable in the clause.

We can combine the PCP theorem with the reduction described above and get the following claim using the following standard proof.

**Claim 6.2** If a 3-SAT instance, with $n$ variables and $m$ clauses, is a “yes” instance, the corresponding CLIQUE instance has a clique of size $m$. If it is a “no” instance, the maximum clique in the corresponding CLIQUE instance has size at most $(1-\epsilon)m$.

**Proof:** A “yes” instance has a satisfying assignment $\pi$. For each clause $C$, we include the unique vertex in $V_C$ corresponding to the restriction of $\pi$ onto the variables in $C$, in set $S$. It is easy to see that $S$ forms a clique of size $m$.

For a “no” instance, suppose there is a clique $S$, in the corresponding CLIQUE instance, of size $\kappa$. Let $\pi$ be any assignment which is an extension of the partial assignments corresponding to the vertices in $S$. Now note that $|S \cap V_C| \leq 1$ for any clause $C$, since $V_C$ is an independent set. Thus there are $\kappa$ clauses $C_1, \ldots, C_\kappa$ such that $|S \cap V_{C_i}| = 1$ for all $1 \leq i \leq \kappa$. From the definition of the reduction, it is easy to see that $\pi$ satisfies all these $\kappa$ clauses. Thus we have $\kappa \leq (1-\epsilon)m$, as desired. \hfill $\square$

We do the following transformation that is a modification of what we did for for SETCOVER. The number of vertices in the CLIQUE instance is $7m$. Let $f(m)$ be any slowly non-decreasing function of $m$ such that $f(m) = \omega(1)$. First, note that we may assume that $m$ is divisible by $f(m)$ without loss of generality. Indeed, we need to add fake clauses to the 3-SAT instance of the type $(x \lor \overline{x} \lor z_1), (x \lor \overline{x} \lor z_2), \ldots$ so that the number of clauses added is at most $f(m)$ and we make $m$ divisible by $f(m)$. Since $f(m)$ is very small compared to $m$, this makes no difference. We create a new CLIQUE instance by introducing a vertex for each subset of size $m/f(m)$ vertices in the old CLIQUE instance. Such a vertex is called a ‘supervertex’. Two supervertices $A, B$, are connected by an edge, if $A \cup B$ is a clique, and $A \cap B = \emptyset$. The last condition, namely, the fact that two sets that are connected must be disjoint is not needed in the SETCOVER reduction, but it is crucial here.

**Claim 6.3** The new instance of the CLIQUE problem has size $2^{o(m)}$.

**Proof:** Using $\binom{n}{k} \leq (ne/k)^k$, we get that the number of supervertices is at most $(7e \cdot f(m))^{7m/f(m)} = \exp(\log(7e \cdot f(m)) \cdot 7m/f(m)) = 2^{o(m)}$, since $f(m) = \omega(1)$. The number of edges in the new CLIQUE instance, being at most the square of the number of vertices, is also $2^{o(m)}$. \hfill $\square$

**Claim 6.4** The maximum clique size in any new instance is exactly $f(m)$. The gap between the clique sizes of the new “yes” and “no” instances is $1/(1-\epsilon)$, which implies $1/(1-\epsilon)$-hardness.

**Proof:** Since the maximum clique size in the old instance is $m$, we get that the maximum clique size in the new instance is $f(m)$. Indeed, we can take the optimum clique and divide it into $m/f(m)$ disjoint sets. By the chosen size these sets are supervertices and their union is the old optimum clique. This shows that the new size of the clique is at least $f(m)$. Since two distinct collection vertices $A$ and $B$ are adjacent in the new instance, only if $A \cup B$ is a clique, and $A, B$ are disjoint, it follows that the largest clique size of the new “yes” instance is exactly $f(m)$ because taking more than $f(m)$ disjoint sets gives a clique of size larger than $m$, contradicting the fact that $m$ is the maximum size of the clique. Thus for a yes instance $f(m)$
is the new size of the maximum clique.

The maximum clique in the new “no” instance, on the other hand, is at most \((1-\epsilon)m/(m/f(m)) = f(m)(1-\epsilon)\), otherwise there would exist a clique in the old instance of size larger than \((1-\epsilon)m\).

The proof is thus complete.

**Claim 6.5** The time can be set to be \(t(\text{OPT}) \cdot n^{O(1)}\) for any non decreasing function \(t\)

**Proof:** Since \(f(m)\) can be as small as we wish, we can make the time \(t(f(m))\) as small as we want. Let \(h(\text{OPT}) = 2^{o(m)}\). Selecting \(f(m) = t^{-1}(h(m))\) gives \(h(m) = 2^{o(m)}\) time. Since \(m,n\) are linearly related here the time can be set to \(2^{o(n)}\) for any \(t\).

### 6.1 A non constant inapproximability

Let the graph that we built in previous subsection (whose optimum for a “yes” instance was \(f(m)\)) be denoted \(H(V,E)\). Recall that its size is:

\[
2^{2\log(7 \cdot e \cdot f(m)) \cdot 7m/f(m)}.
\]

We now recall the power of a graph \(H(V,E)\). We assume the graph is simple, namely has no loops or parallel edges.

**Definition 6.6** The graph \(H^k\) has all the tuples \((v_1, v_2, \ldots, v_k)\) so that any \(v_i\) is a vertex of \(V\). The edges are defined as follows. A tuple \((u_1, u_2, \ldots, u_k)\) is joined to \((v_1, v_2, \ldots, v_k)\) if and only iff for \(i = 1\) to \(k\), either \((u_i, v_i) \in E\) or \(u_i = v_i\).

Note that two different vertices in \(H^k\) have to differ in at least one tuple value.

The following theorem is folklore. Let \(\omega(H)\) be the size of the clique in \(G\).

**Theorem 6.7** \(\omega(H^k) = \omega(H)^k\).

To get a super constant gap we take the graph \(H(V,E)\) of previous section and raise it to the power \(\sqrt{f(m)}\). The choice of \(\sqrt{f(m)}\) is rather arbitrary. Recall that for a “yes” instance \(\omega(G) = m\), with \(m\) the number of clauses in the 3-SAT instance and for a “no” instance \(\omega(G) \leq (1-\epsilon)m\). Hence \(m = \text{OPT}\) for a “yes” instance. Taking this graph to the \(\sqrt{f(\text{OPT})}\) value we get that:

**Corollary 6.8** For \(H(V,E)\sqrt{f(\text{OPT})}\), the value of the clique for a “yes” instance is \(f(\text{OPT})\sqrt{f(\text{OPT})}\) and for a “no” instance at most \((1-\epsilon)\sqrt{f(\text{OPT}) \cdot \text{OPT}\sqrt{f(\text{OPT})}}\).

Note that the new size of the graph is:

\[
2^{2\sqrt{f(m) \log(7 \cdot e \cdot f(m)) \cdot 7m/f(m)}} = 2^{o(m)}.
\]

In addition, the gap is now \(r(\text{OPT}) = (1/(1-\epsilon))\sqrt{f(m)}\). We now describe the gap as a function of the new optimum. The optimum for a “yes” instance is \(\text{OPT}' = f(m)\sqrt{f(m)}\). Thus \((\log \text{OPT}')^{1/3} = \sqrt{f(m)}\). Thus the gap in terms of \(\text{OPT}'\) is: \(r(m) = (1/(1-\epsilon))^{\log^{1/3} \text{OPT}'}\).

**Claim 6.9** Let \(t\) be any non decreasing function and \(r(m) = (1/(1-\epsilon))^{\log^{1/3} \text{OPT}'}\). Then, Clique is \((r,t)\)-FPT-hard.
Proof: The arguments for \( t(\text{opt}) = 2^{\mathcal{O}(m)} \) follows exactly as in Claim 6.5. Because the new optimum for a “yes” instance \( f(m) \sqrt{f(m)} \), can be made arbitrarily small as well.

Also, as the new \( n \) is \( n' = n^{\sqrt{f(m)}} \) we get \( n = n^{\sqrt{f(m)}} \). As \( f(m) \) can be chosen arbitrarily small, and \( n = 7m \), the time as a function of \( n \) is \( n^{\sqrt{f(m)}} \) for any slowly increasing function \( Q(n) \).

7 FPT Hardness for Minimum Maximal Independent Set

In this section we obtain hardness for \textsc{mmis} and thus prove Theorem 4.6.

We start with 3-SAT instance \( I \) with \( m \) clauses and \( q \) variables. We assume that a “yes” instance admits a satisfying assignment and in the case of a “no” instance, any assignment will leave at least one clause unsatisfied. We now describe how to build the new graph \( G(I) = (V(I), E(I)) \).

The building blocks:

1. For every variable \( x \) in \( C \), we define two vertices \( u_x \) and \( \bar{u}_x \). The choice of a vertex \( u_x \) represents an assignment True to \( x \) and the choice of \( \bar{u}_x \) represents a False assignment to \( x \).

2. For every clause we add a set \( W(C) \) of \( q \) copies of the clause. Namely, \( W(C) = \{ w_1^C, \ldots, w_q^C \} \).

Intuitively, we want to create a \textsc{setcover}-like instance in which variables are sets and clauses are elements and a variable \( u_x \) covers \( C \) if \( x \in C \) and \( \bar{u}_x \) covers \( C \) if \( \bar{x} \in C \).

\textbf{Supervertices}: Similar to our construction for \textsc{setcover} and \textsc{clique}, we define a new graph \( H(I) \) with supervertices that are collections of vertices of the type \( u_x, \bar{u}_x \).

Let \( f(q) \) be any slowly increasing function of \( q \) such that \( f(q) = \omega(1) \) and assume, by adding dummy clauses if needed, that \( f(q) \) divides \( q \). The supervertices of \( V(I) \), denoted by \( v_S \), correspond to subsets \( S \subseteq \{ u_x \mid x \in C \} \cup \{ \bar{u}_x \mid x \in C \} \) satisfying the following two conditions:

1. \(|S| = q/f(q)\).

2. \( S \) does not contain both \( u_x, \bar{u}_x \) for any variable \( z \) (i.e., a set \( S \) does not contain a “contradiction” in the truth value assignment).

\textbf{Edges between two supervertices}: Introduce an edge between \( v_{S_1} \) and \( v_{S_2} \) if and only if there exists some variable \( x \) so that either \( u_x \) or \( \bar{u}_x \) belongs to \( S_1 \) and either \( u_x \) or \( \bar{u}_x \) belongs to \( S_2 \). Note that the above gives four cases in which \( v_{S_1}, v_{S_2} \) are connected.

\textbf{Edges between supervertices and \( W(C) \) vertices}: Introduce edges as follows:

1. If a variable \( x \in C \), any supervertex that contains the vertex \( u_x \) is connected to all vertices of \( W(C) \).
2. If a variable \( \bar{x} \in C \), any supervertex that contains \( \bar{u}_x \) is connected to all vertices of \( W(C) \).

**Example:** Say for example \( C = (x \lor \bar{z} \lor w) \). Then any supervertex that contains \( u_x \) is connected to all the copies of \( W(C) \). Also, every supervertex that contains \( \bar{u}_z \) or \( u_w \) is connected to all the copies of \( W(C) \).

**Claim 7.1** Total number of vertices in \( H(I) \) is \( 2^{o(q)}+qm \). The instance \( G(I) \) can be constructed in time \( 2^{o(q)} \).

**Proof:** The total number of vertices in \( H(I) \) of type \( v_S \) for \( S \subset A \) is at most \( \binom{q}{q/f(q)} < (qe/(q/f(q)))^{q/f(q)} < 2^{o(q)} \). Here we again use the inequality \( \binom{q}{k} \leq (qe/k)^k \). The number of vertices of type \( W(C) \) for a clause \( C \) is \( qm \).

**Building an MMIS of size \( f(q) \) for a “yes” instance:**

1. Start with the set \( X = \{ u_x \mid x \text{ is a literal} \} \). This set contains for every variable its vertex copy that corresponds to a True assignment.

2. Decompose \( X \) to \( f(q) \) pairwise disjoint sets each containing \( q/f(q) \) vertices. Let these sets be \( S_1, S_2, \ldots, S_{q/f(q)} \). We want to derive sets so that \( v_{S} \) is a feasible MMIS, which is of course not the case so far (because not all \( W(C) \) are covered).

3. We now modify sets \( S_i \) to obtain sets \( T_i \) as follows. Fix a satisfying assignment \( \tau \) to the variables. We start by setting \( T_i = S_i \) for all \( i \). If \( \tau(x) \) is False, then for the unique \( i \) so that \( u_x \in T_i \), remove \( u_x \) from \( T_i \) and add \( \bar{u}_x \) to \( T_i \). This is done for all variables. The final \( T_i \) sets are called the assignment sets. Our solution will be \( \mathcal{I} = \{ v_{T_i} \mid T_i \text{ is an assignment set} \} \).

**Claim 7.2** The set \( \{ v_{T_i} \} \) is independent in \( H(I) \).

**Proof:** For the vertices \( v_{T_i}, v_{T_j} \) with \( i \neq j \) to be connected it must be that some \( x \) so that either \( u_x \) or \( \bar{u}_x \) belongs to \( T_i \) and either \( u_x \) or \( \bar{u}_x \) belongs to \( T_j \). Clearly, this implies that \( u_x \in S_i \cap S_j \). This is a contradiction to the fact that the sets \( \{ S_p \} \) are pairwise disjoint.

**Claim 7.3** The \( f(q) \) vertices \( \{ v_{T_i} \} \) defined above form a dominating set in \( H(I) \).

**Proof:** We first show each vertex in \( W(C) \) is adjacent to some vertex \( v_{T_i} \). Note that \( \tau \) satisfies all clauses \( C \). One possibility is that \( \tau(x) \) is True and \( x \in C \). Thus the unique assignment set \( T_i \) that contains \( u_x \) is connected to all the copies \( W(C) \) of \( C \). Alternatively, if \( \tau(x) \) is False and \( \bar{x} \in C \), the unique \( T_i \) that contains \( \bar{u}_x \) is connected to all copies of \( W(C) \).

We now show that \( \mathcal{I} \) dominates every supervertex not in \( \mathcal{I} \). Let \( v_S \) be a vertex of \( H(I) \) that does not belong to \( \mathcal{I} \). Pick an arbitrary variable \( x \) so that either \( u_x \in S \), or \( \bar{u}_x \in S \). By construction there is some assignment set \( T_i \in \mathcal{I} \) that contains \( u_x \) or \( \bar{u}_x \). In all the four cases above, by definition, there is an edge between \( v_{T_i} \) and \( v_S \).

Thus we just proved the following corollary.

**Corollary 7.4** The “yes” instance admits a solution of size \( f(q) \).

**Claim 7.5** For a no instance the minimum MMIS is of size larger than \( q \).
Proof: Let \( S \) be the optimum MMIS of the “no” instance. Note that all super vertices chosen by the optimum have to be consistent. Namely, we can not have \( u_x \) belonging to one set \( T_i \) in \( S \) and \( \bar{u}_x \) to some \( T_j \in S \) because this will imply an edge between \( v_{T_i} \) and \( v_{T_j} \) and a contradiction. In particular, this implies that vertices \( \{ v_{T_i} \} \) represent a (maybe partial) truth assignment to the variables.

Since we are dealing with a no instance, there must be a clause \( C \) that is not satisfied by this partial assignment. This means that none of the vertices that correspond to literals that satisfy \( C \) are in any set of \( S \). For example if \( C = (x \lor \bar{z} \lor w) \) then there may be one set related to \( x \) but it contains \( \bar{u}_x \), because the assignment does not satisfy \( C \). There may be one set related to \( z \), but it contains \( u_z \), and there may be a set for \( w \), but it contains \( \bar{u}_w \). This means that the \( q \) copies \( W(C) \) must be present in \( S \), since it is a maximal independent set. Thus the size of \( S \) is at least \( q \).

\[ \square \]

**Theorem 7.6** Assuming the ETH, MMIS problem is \((r, t)\)-hardness, for any \( r, t \).

Proof: Since the new optimum for a yes instance is \( f(q) \) where \( f \) is an arbitrarily slow growing function. For any given functions \( r \) and \( t \), we can make sure that \( r(f(q)) < q/f(q) \) and \( t(f(q)) = 2^{o(q)} \).

Note that by Claims 7.3 and 7.5, the gap between “yes” and “no” instances is larger than \( q/f(q) \). If there existed an \((r, t)\)-FPT-approximation for MMIS, we could distinguish between a “yes” and a “no” instance of 3-SAT in time \( 2^{o(q)} \), contradicting the ETH.

\[ \square \]

**8 Open problems**

We can look for more problems that are \((r, t)\)-FPT-hardness. Given a problem we first should check if there is no easy \( g(\text{opt}) \) approximation. many such problems are presented in [5]. All the problems in question were highly inapproximable and \( W[1] \)-hard Since we believe that such an approximation is not possible for CLIQUE we ask:

**Open Problem 1:** Which \( W[1] \) optimization problems are \((r, t)\)-FPT-hard for various functions \( r \) and \( t \)?

One interesting thing is that all problems presented in [5] were only \( W[1] \)-hard.

**Open Problem 2:** Is there an \( f(\text{opt}) \) approximation for any \( W[2] \)-hard problem, that runs in polynomial or FPT time?

It may be hard to prove that such a thing does not exist as it will imply \( W[1] \neq W[2] \). Still, this suggests an indirect way to try and prove this important and widely believed conjecture.

Another way to make the optimum smaller is by taking a random sample. This technique fails miserably for SETCOVER, but also for CLIQUE for which we had hopes that this will succeed.

**Open problem 3:** Is there a \( W[1] \) or \( W[2] \) hard problem, so that we can decrease \( \text{OPT} \) by random sampling, keep a large enough gap, and imply a super-exponential hardness in \( \text{OPT} \) due to the optimum becoming smaller?
References


