

# Approximating Fault-Tolerant Group-Steiner Problems

Rohit Khandekar\*      Guy Kortsarz †      Zeev Nutov‡

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## Abstract

In this paper, we initiate the study of designing approximation algorithms for **Fault-Tolerant Group-Steiner (FTGS)** problems. The motivation is to protect the well-studied group-Steiner networks from edge or vertex failures. In **Fault-Tolerant Group-Steiner** problems, we are given a graph with edge- (or vertex-) costs, a root vertex, and a collection of subsets of vertices called groups. The objective is to find a minimum-cost subgraph that has two edge- (or vertex-) disjoint paths from each group to the root. We present approximation algorithms and hardness results for several variants of this basic problem, e.g., edge-costs vs. vertex-costs, edge-connectivity vs. vertex-connectivity, and 2-connecting from each group a single vertex vs. two distinct vertices from each group. Main contributions of our paper include the introduction of very general structural lemmas on connectivity and a charging scheme that may find more applications in the future. Our algorithmic results are supplemented by inapproximability results, which are tight in some cases.

Our algorithms employ a variety of techniques. For the edge-connectivity variant, we use a primal-dual based algorithm for covering an *uncrossable* set-family, while for the vertex-connectivity version, we prove a new graph-theoretic lemma that shows equivalence between obtaining two vertex-disjoint paths from two vertices and 2-connecting a carefully chosen single vertex. To handle large group-sizes, we use a  $p$ -Steiner tree algorithm to identify the “correct” pair of terminals from each group to be connected to the root. We also use a non-trivial charging scheme to improve the approximation ratio for the most general problem we consider.

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\*IBM T.J.Watson Research Center. Email: rohitk@us.ibm.com

†Rutgers University, Camden. Email: guyk@camden.rutgers.edu Partially supported by NSF grant 08129959

‡The Open University of Israel. Email: nutov@openu.ac.il

# 1 Introduction

The fault-tolerant network design problems are well-studied in the theory of combinatorial optimization and approximation algorithms. The basic goal in these problems is to design a minimum-cost network that satisfies some prescribed connectivity requirements. Higher connectivity requirements are usually enforced for fault-tolerance — in order to protect connectivity in the solution against edge or vertex failures. A well-studied fault-tolerant network design problem is the **Steiner Network** problem. In this problem, we are given edge- (or vertex-) connectivity requirement  $r_{ij}$  between every pair  $\{i, j\}$  of vertices, and the goal is to design a network with at least  $r_{ij}$  edge- or vertex-disjoint paths between  $i$  and  $j$  for each pair  $\{i, j\}$ .

In this paper, we study fault-tolerant versions of the **Group-Steiner** problem. In this problem, we are given a (undirected or directed) graph  $G = (V, E)$  with edge- or vertex- costs, a root vertex  $r \in V$ , and a collection of subsets (groups)  $\mathcal{S} = \{S_1, \dots, S_q\}$  of  $V \setminus \{r\}$ . The objective is to find a minimum cost subgraph  $H$  of  $G$  that contains two edge- or vertex-disjoint paths from each group  $S_i$  to the root. This problem models the flexibility, often arising in problems in VLSI design [26], in connecting any vertices from a given group as well as the fault-tolerance which requires the solution to be robust under edge- or vertex-failures.

## 1.1 Previous work on fault-tolerance problems

Fault tolerance problems have been extensively studied in Combinatorial Optimization and Approximation Algorithms. Consider for example the well known **Steiner Network** problem. Given an undirected graph  $G = (V, E)$  with edge-costs  $\{c_e \mid e \in E\}$ , and requirement  $r_{ij}$  for every pair of vertices  $i, j \in V$ , the goal is to find a minimum cost subgraph  $H$  of  $G$  that contains at least  $r_{ij}$  edge-disjoint paths between  $i$  and  $j$ , for all  $i, j$ . The network  $H$  is fault tolerant in the sense that a pair  $i, j$  can sustain  $r_{ij} - 1$  link failures and still be connected. The best ratio known for this problem is 2 due to Jain [15].

The internally-disjoint path version of the **Steiner Network** problem is very hard to approximate [17, 22, 9, 19]. The currently best known ratio for this problem is  $O(k^3 \log n)$  for edge-costs due to Chuzhoy and Khanna [8] and  $O(k^4 \log^2 n)$  for vertex-costs [21]. The rooted version, where a root  $s$  is given and  $r_{ij} = 0$  for all pairs  $i, j$  so that  $i \neq s$  and  $j \neq s$ , has gotten a significant attention recently on the high level [3, 5, 9, 24, 8, 21]. This problem is at least as hard to approximate as **Directed Steiner Tree** [19]. The best approximation ratio known for this problem for general rooted demands is  $O(k^2)$  for edge-costs and  $O(k^2 \log n)$  for vertex-costs [21], where  $k = \max r_{ij}$ . We mention that prior to the work of [21], a randomized approximation algorithm with ratio  $k^{O(k^2)} \log^4 n$  was developed by Chakraborty, Chuzhoy, & Khanna [3], then improved to  $k^{O(k)} \log n$  by Chekuri & Korula [5], and then improved to  $O(k^2 \log n)$  by Chuzhoy & Khanna [9, 8] and [24]. Note that  $k$  can be as large as  $\Omega(n)$ .

A particular case of fault tolerance problems with 2 disjoint paths has also received an attention on the high level [1, 20, 5]. Lau et al. [20] presented an  $O(\log^2 n)$ -approximation algorithm for the problem of finding a minimum-cost 2-edge connected subgraph with *at least*  $k$  vertices. This problem can be seen as a fault-tolerant generalization of the  $k$ -MST problem which requires to find a minimum spanning tree on  $k$  vertices. The best known approximation ratio for the  $k$ -MST problem is 2 [11]. Chekuri and Korula [6] presented an  $O(\log^2 n)$ -approximation for the problem

of finding a minimum-cost 2-vertex connected subgraph with at least  $k$  terminals. In [1, 5], the fault-tolerance version of the Buy-at-Bulk problem was studied, where two edge-disjoint paths are required to be included from every terminal to the root.

In the same spirit, in this paper we consider a generalization of the **Group-Steiner Tree** problem. In the usual **Group-Steiner Tree** problem, we are given a graph  $G = (V, E)$ , edge costs  $\{c_e : e \in E\}$ , a root  $r \in V$ , and a collection of subsets (groups)  $\mathcal{S} = \{S_1, \dots, S_q\}$  of  $V \setminus \{r\}$ . The objective is to find a minimum cost subtree  $T$  of  $G$  that contains  $r$  and at least one vertex from each group  $S_i \in \mathcal{S}$ . The best known approximation ratio for this problem is  $O(\log^3 n)$  [13]. The **Fault-Tolerant Group-Steiner Tree** problem, on the other hand, requires obtaining two (edge- or vertex-) disjoint paths between each group to the root. We are not aware of any previous work on **Fault-Tolerant Group-Steiner** problems.

## 1.2 Problem variants studied in this paper

One can define several variants of the **Fault-Tolerant Group-Steiner** problem, based on whether we desire 2-edge- or 2-vertex-connectivity, whether we have edge- or vertex-costs, whether we wish to 2-connect to the root a single vertex from each group or two distinct vertices from each group, etc. Below we formally define all the variants studied in this paper. Two paths are said to be *internally-disjoint* if they are vertex-disjoint except for their end-points. Each of the following problems takes a graph  $G = (V, E)$  on  $n$  vertices with edge-costs  $\{c_e \mid e \in E\}$  (or with node-costs  $\{c_v \mid v \in V\}$ ), a root  $r \in V$ , and groups  $\mathcal{S} = \{S_1, \dots, S_q\}$  as input, and is required to compute a min-cost subgraph  $H$  of  $G$  with at least two edge/vertex-disjoint paths from each group  $S_i$  to the root, so that the endnodes of these paths are distinct. Unless stated otherwise, we consider the edge-cost version and assume that  $G$  is undirected. We also assume that the groups  $S_1, \dots, S_q$  are pairwise disjoint.<sup>1</sup> The vertices in the groups  $S_i$  are called *terminals*. We add the prefix **EC-** for edge-connectivity and the prefix **VC-** for vertex-connectivity. We add the suffix “ $-k$ ” after the name of the problem if the instances are restricted to satisfy  $|S_i| \leq k$  for all  $i = 1, \dots, q$ .

- **EC-FTGS**: Here for every  $i = 1, \dots, q$ ,  $H$  should contain at least two edge-disjoint  $S_i r$ -paths; the end-points in  $S_i$  of these paths should be distinct.
- **VC-FTGS**: The same as **EC-FTGS**, except that the paths should be internally-disjoint.
- **EC-FTGS- $k$**  and **VC-FTGS- $k$** : These are **EC-FTGS** and **VC-FTGS**, respectively, restricted to instances with  $|S_i| \leq k$  for all  $i = 1, \dots, q$ .

In the edge-connectivity case, for both edge and vertex-costs, the version when the end-points in  $S_i$  of the two  $S_i r$ -paths may or may not be distinct, is easily reduced to **EC-FTGS** as follows. For every terminal  $s$ , add a new vertex  $s'$  of cost 0 connected to  $s$  with an edge of cost 0, and add  $s'$  to every group  $S \in \mathcal{S}$  that contains  $s$ . After this transformation, we can assume, without loss of generality, that the two edge-disjoint paths from each group start from *distinct* terminals in

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<sup>1</sup>This can be typically assumed by making multiple copies of the vertices in multiple groups and adding zero-cost edges connecting the different copies. This reduction, however, increases the number of vertices in the graph, thus possibly affecting the approximation ratio.

that group. This may double the number of vertices, and cause a constant loss in approximation ratios that depend on  $n$ .

Because the edge  $s'$ 's must be used this transformation holds only in the edge-disjoint case.<sup>2</sup> For the reduction from a single terminal to two terminals in the vertex case we make a copy of every node  $s'$  for every node  $s \in S$ . We connect  $s'$  to the same neighbors of  $s$ . Clearly there are two node disjoint paths from  $s$  if there are two disjoint node paths from  $s, s'$  in the new graph.

We also consider the version when we insist that a single vertex from each group must be 2-edge-connected to the root. Namely, for every  $i = 1, \dots, q$  there should exist a vertex  $v_i \in S_i$  such that  $H$  contains 2 edge-disjoint  $rv_i$ -paths. We call this version **Single EC-FTGS**.

### 1.3 The difficulty in approximating Fault-Tolerant problems

When two disjoint paths from every group to the root are required, we cannot use the standard transformation to Bartal trees [2, 10] as done in the approximation of the **Group Steiner** problem [13]. This is so because disjoint paths from a group to the root in a Bartal tree do not necessarily correspond to disjoint paths in the original graph.

We now give a strong evidence that an approximation that is polylogarithmic in  $n$  for either **EC-FTGS- $k$**  and **VC-FTGS- $k$**  may be very hard to obtain as it implies solving a long standing open problem. Note that we can reduce the usual **Group Steiner** problem to **EC-FTGS- $k$**  or **VC-FTGS- $k$**  problems by adding a new vertex, which is connected to the root by a zero-cost edge, to each group. Since we get one path for “free”, any solution to **EC-FTGS- $k$**  and **VC-FTGS- $k$**  corresponds to a solution of the usual **Group Steiner** problem and vice-versa.

Now since an algorithm for **EC-FTGS- $k$**  or **VC-FTGS- $k$**  cannot use Bartal trees, it must solve the **Group Steiner** problem as well **without** using Bartal trees. The question if **Group Steiner** problem can be approximated within a polylogarithmic ratio without first reducing the graph into trees is a long standing open question and seemingly a very hard one.

To the best of our knowledge, the best known ratio for **Group Steiner** problem without using Bartal trees is  $O(n^\epsilon)$  for any constant  $\epsilon > 0$ , with running time  $n^{f(1/\epsilon)}$ . The *recursive greedy* technique [28, 18, 4], used in this algorithm, is a complex greedy approach with quite delicate analysis that seems completely inappropriate for the requirement of two disjoint paths. Thus even an  $n^\epsilon$  approximation for every universal constant  $\epsilon$  seems to be a quite significant challenge for our problems in the current state of knowledge and techniques.

In our opinion, this is a strong evidence that it is quite a challenge to get polylogarithmic ratios for either **EC-FTGS- $k$**  or **VC-FTGS- $k$**  in polynomial time.

**Remark:** The above simple reduction shows that the  $\Omega(\log^{2-\epsilon} n)$  hardness for the **Group Steiner** problem applies also for **EC-FTGS- $k$**  and **VC-FTGS- $k$** . However, from the above evidence, **EC-FTGS- $k$**  and **VC-FTGS- $k$**  may in fact be much harder to approximate than  $\Omega(\log^{2-\epsilon} n)$ , and it may be that a polynomial ratio is the best we can hope for.

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<sup>2</sup>is the following reduction correct?

Problem	Edge-Connectivity (EC)	Vertex-Connectivity (VC)
FTGS-2	3.55, Vertex-Cover-hard $O(\log n)$ , $\Omega(\log n)$ -hard (vertex-costs)	$O(\log^2 n)$
FTGS- $k$	$O(k \log^2 n)$	$O(k \log^2 n)$
FTGS	$O(\sqrt{n} \log n)$ Group Steiner Tree-hard	$O(\sqrt{n} \log n)$ Group Steiner Tree-hard
Single FTGS	Label Cover-hard (directed)	Label Cover-hard (directed)

Table 1: Approximation ratios and hardness results for FTGS variants. The extra assumptions, if any, are given in the parentheses.

## 1.4 Our results

We start with a definition. For two optimization problems  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , we say that  $\mathcal{P}_1$  is  $\mathcal{P}_2$ -hard if existence of a polynomial time  $\rho$ -approximation algorithm for  $\mathcal{P}_1$  implies existence of a polynomial time  $\rho$ -approximation for  $\mathcal{P}_2$ . Similarly,  $\mathcal{P}_1$  is  $\Omega(f(n))$ -hard if there exists a constant  $\epsilon > 0$  so that  $\mathcal{P}_1$  admits no  $\epsilon f(n)$ -approximation algorithm, unless  $P=NP$ .

Our main results are summarized in Theorem 1.1 and in Table 1.

### Theorem 1.1

- (i) EC-FTGS-2 admits the following approximation ratios:  
 $(2 + \gamma)$  for edge costs, where  $\gamma < 1.55$  is the best ratio for the Steiner Tree problem, and  $O(\log n)$  for vertex costs. Moreover, the edge-cost version is Vertex-Cover-hard and the vertex-cost version is  $\Omega(\log n)$ -hard.
- (ii) EC-FTGS- $k$  and VC-FTGS- $k$  admit an  $O(k \log^2 n)$ -approximation algorithm. EC-FTGS and VC-FTGS admit an  $O(\sqrt{n} \log n)$ -approximation algorithm. EC-FTGS and VC-FTGS are Group Steiner Tree-hard and thus are  $O(\log^{2-\epsilon} n)$ -hard for any constant  $\epsilon > 0$ .
- (iii) The directed version of Single EC-FTGS problem admits no  $2^{\log^{1-\epsilon} n}$ -approximation for any constant  $\epsilon > 0$ , unless  $NP \subseteq DTIME(n^{\text{polylog}(n)})$ .

The results (i), (ii) and (iii) are proved in Sections 2 and 3 and 4, respectively.

## 1.5 A summary of our techniques

In Part (i), we observe that *all* terminals need to be connected to the root, thus our algorithm first computes a Steiner tree connecting all the terminals to the root (not necessarily by disjoint paths). We then note that the residual problem can be reduced to the one of covering an *uncrossable* set-family by edges. In the case of edge-costs, this problem can be solved using the primal-dual method of Goemans et al. [14], or via the iterative rounding technique of Jain [15]. Overall we obtain a  $(2+\gamma)$ -approximation for this problem if we use a  $\gamma$ -approximation algorithm for Steiner Tree as a subroutine. Since  $\gamma < 1.55$  is known [27], we get a 3.55-approximation. A similar approach works for vertex-costs using the  $O(\log n)$ -approximation algorithm of [16] for

computing the Steiner tree, and using in the augmentation step the algorithm of [23] instead of that of [14].

Our main technical contribution is in our algorithms for problems in Part (ii). Our algorithm for VC-FTGS-2 also reduces the problem to an augmentation problem, by first computing a Steiner tree connecting all the terminals to the root. To solve this augmentation problem, we introduce an interesting graph theoretic lemma (Lemma 3.4). Our lemma provides an equivalence between 2-connecting two vertices to the root and 2-connecting a single carefully chosen vertex to the root. We believe that this lemma may be of independent interest. For every such vertex  $v$ , we let the profit  $p(v)$  of  $v$  be the number of groups that will be satisfied if  $v$  is 2-connected to the root. The *density* version of the residual problem seeks a subgraph that minimizes the ratio of its cost over the total profit of vertices that are 2-vertex-connected to the root. We then use the  $O(\log n)$ -approximation algorithm by Chekuri and Korula [6] for this density problem, combined with the greedy method.

Our algorithms for VC-FTGS- $k$  derives the intuition from that for VC-FTGS-2, and also uses other techniques. Let us call a pair of terminals from a group that are connected to the root via vertex-disjoint paths a *twin pair* (see Definition 3.1). We divide the progress of our algorithm into logarithmically many phases. In each phase, our algorithm first tries to connect the twin pairs from at least half of the remaining groups to the root (not necessarily by vertex-disjoint paths). Unfortunately, we do not know the twin pairs a-priori. Nevertheless, we show that one can use several applications of the  $p$ -Steiner tree algorithm [7] to obtain a relatively low-cost tree that achieves this. The residual problem is now similar to the augmentation problem considered for the case of VC-FTGS-2. We once again use our graph theoretical lemma (Lemma 3.4), relate 2-connecting a twin pair to 2-connecting a carefully chosen vertex, define  $p(v)$  of such a vertex as in the VC-FTGS-2 algorithm. It turns out that a naive application and analysis of the greedy method as in our VC-FTGS-2 algorithm gives an approximation ratio of  $O(k^2 \log^2 n)$ , since there could be  $\Omega(k^2)$  pairs that may claim profit for a single group. An important component in our analysis is a careful counting of how the profit can be claimed to prove an overall approximation factor of  $O(k \log^2 n)$ .

The last statement in Part (ii) of Theorem 1.1 then easily follows. To “cover” small groups, i.e., groups of size at most  $k = \sqrt{n}/\log n$ , we use the algorithm above. Since the groups are disjoint, the number of large groups is at most  $n/k = \sqrt{n} \log n$ . We cover the large groups one at a time. Overall, the approximation ratio is  $O(\min\{k \log^2 n, n/k\}) = O(\sqrt{n} \log n)$ .

The proof of part (iii) follows from a reduction from the Min-Rep problem.

## 2 Proof of Part (i)

We start with some definitions. An edge  $e$  is said to *cover* a subset  $X \subset V$  of vertices if exactly one end-point of  $e$  lies in  $X$ . Let  $\mathcal{F}$  be a collection of subsets of  $V$ . We say that an edge-set  $E'$  covers  $\mathcal{F}$  if for each  $X \in \mathcal{F}$ , there is an edge  $e \in E'$  that covers  $X$ . The **Set-Family Edge-Cover** problem with edge-cost is to find a minimum-cost collection of edges  $E'$  that covers  $\mathcal{F}$ . In the vertex-cost version, we wish to minimize the total cost of vertices incident to  $E'$  that covers  $\mathcal{F}$ . The family  $\mathcal{F}$  may not be given explicitly, but we require that certain queries related to  $\mathcal{F}$  can be answered in polynomial time. Specifically, we assume that, in the edge-cost version, for any

edge-set  $E'$ , the inclusion minimal members of  $\mathcal{F}$  that are *not* covered by  $E'$  can be computed in polynomial time; while, in the vertex-cost version, for any  $s, t \in V$ , one can compute in polynomial time a min-cost cover of all members of  $\mathcal{F}$  that separate  $s$  and  $t$ .

We call a family  $\mathcal{F}$  of sets *uncrossable* if  $X \cap Y, X \cup Y \in \mathcal{F}$  or  $X \setminus Y, Y \setminus X \in \mathcal{F}$  for any  $X, Y \in \mathcal{F}$ . Our algorithms for EC-FTGS-2 problem with edge-costs or vertex-costs use the following results, respectively.

**Theorem 2.1 (Goemans et al. [14])** *The Set-Family Edge-Cover problem with edge-costs and with uncrossable set-family  $\mathcal{F}$  admits a 2-approximation algorithm.*

**Theorem 2.2 ([23])** *The Set-Family Edge-Cover problem with vertex-costs and with uncrossable set-family  $\mathcal{F}$  admits an  $O(\log n)$ -approximation algorithm.*

## 2.1 Algorithmic results

Since the problem insists that the two edge-disjoint paths from each group must start at distinct terminals in the group, the optimum solution contains a Steiner tree containing all the terminals and the root. Our algorithm first finds a Steiner tree  $T$  that connects all the terminals to the root. If we use an  $\alpha$ -approximation algorithm for this step ( $\alpha < 1.55$  for the edge-costs [27] and  $\alpha = O(\log n)$  for the vertex-costs [16]), we get  $\text{COST}(T) \leq \alpha \cdot \text{OPT}$  where  $\text{OPT}$  denotes the cost of the optimum solution.

For  $X \subset V$ , let  $\text{deg}_T(X)$  denote the number of edges in  $T$  from  $X$  to  $V \setminus X$ . Define an instance of Set-Family Edge-Cover by setting

$$\mathcal{F} = \{X \subseteq V \setminus \{r\} \mid \text{deg}_T(X) = 1, S \subseteq X \text{ for some } S \in \mathcal{S}\}.$$

We now present two important observations.

**Lemma 2.3** *For  $I \subseteq E \setminus E(T)$ , the set  $T \cup I$  is a feasible solution to EC-FTGS-2 if, and only if,  $I$  covers  $\mathcal{F}$ .*

*Proof:* Note that  $T$  has a path from each terminal to the root. Thus by Menger's Theorem,  $H = T \cup I$  is a feasible solution to EC-FTGS-2 if, and only if,  $\text{deg}_H(X) \geq 2$  for every set  $X \subseteq V \setminus \{r\}$  that contains some group  $S \in \mathcal{S}$ . As  $\text{deg}_T(X) \geq 1$  for any  $X \subseteq V \setminus \{r\}$  that contains at least one vertex from some group, we obtain that the latter condition is equivalent to  $\text{deg}_I(X) \geq 1$  for every  $X \in \mathcal{F}$ . ■

**Lemma 2.4** *The set family  $\mathcal{F}$  is uncrossable.*

*Proof:* Note that by the definition of  $\mathcal{F}$ ,  $X \in \mathcal{F}$  if, and only if,  $X$  is a union of a rooted proper subtree of  $T$  that contains a group  $S \in \mathcal{S}$  and some subset of vertices not in  $T$ . Let  $X, Y \in \mathcal{F}$ . Then  $X \cap T, Y \cap T$  are disjoint, or one of them contains the other. In the former case, we have  $X \setminus Y, Y \setminus X \in \mathcal{F}$ ; e.g.,  $X \setminus Y \in \mathcal{F}$  since  $(X \setminus Y) \cap T = X \cap T$ , hence  $X \setminus Y$  is a union of the subtree contained in  $X$  and the vertex subset  $X \setminus (T \cup Y)$  disjoint to  $T$ . In the latter case,  $X \cap Y, X \cup Y \in \mathcal{F}$ ; e.g., if  $X \subseteq Y$ , then  $X \cap Y$  is a union of the subtree contained in  $X$  and some

vertices not in  $T$ , while  $X \cup Y$  is the union of the subtree contained in  $Y$  and some vertices not in  $T$ .  $\blacksquare$

It is easy to check that for any edge set  $I \subset E \setminus E(T)$ , the minimal members of the family  $\mathcal{F}$  not covered by  $I$  can be computed in polynomial time. Moreover, for any  $s, t \in V$ , a min-cost cover of all members in  $\mathcal{F}$  that separate  $s$  and  $t$  can also be computed in polynomial time. Thus, we can use the algorithms in Theorems 2.1 and 2.2 respectively to complete the solutions for the edge- and vertex-cost versions.

## 2.2 Hardness of approximation results

We now show that EC-EFTGS-2 is Vertex-Cover hard in the case of edge-costs, and that it is Hitting-Set-hard, i.e.,  $\Omega(\log n)$ -hard, in the case of vertex-costs.

Let  $J = (V_J, E_J)$  be an instance of Vertex-Cover. Define an instance  $\{G = (V, E), \{c_e : e \in E\}, r, \mathcal{S}\}$  of 2-EC-FTGS-2 as follows. Set  $V = V_J \cup \{a, r\}$ , connect every vertex in  $V_J$  to  $a$  with an edge of cost 0 and connect  $a$  to  $r$  with an edge of cost 0. Let  $T$  denote the set of these zero-cost edges. Connect each vertex in  $V_J$  to  $r$  with an edge of cost 1 each. The set  $\mathcal{S}$  of pairs is defined by edges of  $E_J$ , namely,  $\mathcal{S} = \{\{u, v\} \mid (u, v) \in E_J\}$ . Note that the optimum solution, without loss of generality, picks all the edges in  $T$ . It is now easy to see that  $T + H$  is a feasible solution to the obtained instance of EC-FTGS-2 if and only if the set of end-points of the edges in  $H$  is a vertex-cover in  $J$ .

In the case of vertex-costs, EC-FTGS-2 is easily reduced to the Steiner Tree problem with vertex-costs which is known to be Hitting-Set-hard [16]. Given an instance  $\{J = (V_J, E_J), r, S\}$  of Steiner Tree with vertex-costs, for every  $s \in S$  add a copy  $s'$  of cost 0 and connect  $s'$  to  $r$ . The set of pairs is  $\mathcal{S} = \{\{s, s'\} \mid s \in S\}$ . It is easy to see that  $T$  is a feasible solution to Steiner Tree with vertex-costs if and only if  $T \cup \{(r, s') \mid s \in S\}$  is a feasible solution to the constructed 2-EC-FTGS-2 instance.

The proof of Part (i) of Theorem 1.1 is thus complete.

## 3 Proof of Part (ii)

### 3.1 Algorithm for VC-FTGS-2

In this section, we introduce our main technical ideas. We present an  $O(\log^2 n)$ -approximation algorithm for VC-FTGS-2.

As in the edge-connectivity case, we first compute a Steiner tree  $T$  of cost  $\text{COST}(T) \leq \alpha \cdot \text{OPT}$  connecting all terminals to the root. We have  $\alpha < 1.55$  for edge-costs and  $\alpha = O(\log n)$  for vertex-costs. Our algorithm uses set-cover like approach in which we iteratively add partial solutions with low *density*, i.e., low cost to profit ratio, till we complete the solution. We get one logarithmic factor in the approximation from the set-cover analysis (and since the number of groups is  $O(n)$ ) and another logarithmic factor from the fact that we can only compute  $O(\log n)$  approximation to the minimum-density subproblem.

Given a partial solution  $I \subset E \setminus E(T)$ , let the *deficiency* of  $I$  be the number of groups in  $\mathcal{S}$  that are not 2-vertex-connected to  $r$  in  $T \cup I$ . Let the *density* of an edge set  $F \subset E \setminus (E(T) \cup I)$

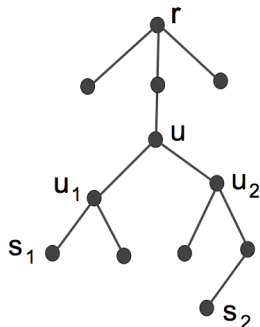


Figure 1: An example of converting the instance into one satisfying Property P.

be  $\text{COST}(F)$  divided by the decrease in the deficiency caused by adding  $F$  to  $T \cup I$ . The following two lemmas captures the essence of our algorithm for VC-FTGS-2.

**Lemma 3.1** *Given a partial solution  $T \cup I$ , the problem of computing a minimum density augmenting edge set  $F \subset E \setminus (E(T) \cup I)$  for VC-FTGS-2 admits an  $O(\log n)$ -approximation algorithm.*

**Lemma 3.2** *The algorithm in Lemma 3.1 can be used to obtain  $O(\log^2 n)$  approximation for VC-FTGS-2.*

As mentioned, the proof of Lemma 3.2 follows from the standard set-cover like analysis and is omitted. In the rest of this section we prove Lemma 3.1. We ignore the groups in  $\mathcal{S}$  that are already connected in  $T \cup I$  to the root via 2 vertex-disjoint paths starting from distinct vertices.

We start by recalling some definitions. A vertex  $v \in V$  is a cut-vertex of a graph  $H$  if  $H \setminus \{v\}$  has more connected components than  $H$ . A cut-vertex  $v$  of  $H$  is said to *separate* vertex  $r$  and set  $S \subset V \setminus \{r, v\}$  if  $r$  and  $S$  belong to the same connected component of  $H$  but  $H \setminus \{v\}$  does not contain a path from  $r$  to any vertex in  $S$ . Consider a group  $\{s_1, s_2\} \in \mathcal{S}$ . By Menger's Theorem we have:

**Proposition 3.1** *A subgraph that contains an  $rs_1$ -path and an  $rs_2$ -path, contains such paths that are internally vertex disjoint if and only if it has no cut-vertex that separates  $r$  and  $\{s_1, s_2\}$ .*

Now think of the tree  $T$  as being rooted at  $r$ . For any two vertices  $s_1, s_2 \in T$ , we define  $\text{LCA}(s_1, s_2)$  to be the least common ancestor of  $s_1$  and  $s_2$  in  $T$ . Consider  $S = \{s_1, s_2\} \in \mathcal{S}$  with  $u = \text{LCA}(s_1, s_2)$  in  $T$ . Note that  $u \neq r$  since  $S$  is not 2-vertex connected to  $r$  via paths starting from  $s_1$  and  $s_2$ . Let  $\mathcal{U}(S) = \{u_1, u_2\}$  be two (possibly identical) vertices defined as follows.

- If  $u \notin \{s_1, s_2\}$ , then let  $u_1$  (resp.  $u_2$ ) be the child of  $u$  that lies on the  $rs_1$ - (resp.  $rs_2$ -) path in  $T$ .
- If  $u \in \{s_1, s_2\}$ , then let  $u_1, u_2 = u$ .

Define a family  $\mathcal{U} = \mathcal{U}(\mathcal{S}, T)$  of pairs (groups) as  $\mathcal{U} = \{\mathcal{U}(S) \mid S \in \mathcal{S}\}$ . Note that the pairs in  $\mathcal{U}$  may not be disjoint, and that some “pairs” in  $\mathcal{U}$  are in fact singletons. The following lemma captures the “equivalence” between covering pairs in  $\mathcal{S}$  and pairs in  $\mathcal{U}$ .

**Lemma 3.3** *For any edge set  $F \subset E \setminus (E(T) \cup I)$ , the solution  $H = T \cup I \cup F$  contains 2-vertex disjoint paths (with distinct starting points) between  $r$  and  $S$  if, and only if, it contains 2-vertex disjoint paths (with distinct starting points) between  $r$  and  $\mathcal{U}(S)$ .*

*Proof:* Let  $u = \text{LCA}(s_1, s_2)$  where  $S = \{s_1, s_2\}$ . By Proposition 3.1,  $S$  is 2 vertex-connected to  $r$  in  $H = T \cup I \cup F$  if and only if  $H$  has no cut-vertex separating  $S$  from  $r$ ; namely, no vertex on the  $ur$ -path in  $T$  is a cut-vertex of  $H$ . By the definition of  $\mathcal{U}(S) = \{u_1, u_2\}$  and Proposition 3.1, this is equivalent to the property that  $\mathcal{U}(S)$  is 2-connected to  $r$  in  $H$ . ■

The above lemma implies that the densities of  $F$  w.r.t.  $\mathcal{S}$  and w.r.t.  $\mathcal{U}$  are equal. Furthermore,  $T \cup I \cup F$  is a feasible solution w.r.t.  $\mathcal{S}$  if and only if it is w.r.t.  $\mathcal{U}$ . Note that the groups  $\mathcal{U}$  satisfy a special property, which is crucial in rest of the analysis.

**Property P:** For all groups  $\{u_1, u_2\} \in \mathcal{U}$ , either  $u_1 = u_2$  or  $u_1, u_2$  have the same parent in  $T$ .

**Lemma 3.4** *Let  $U = \{u_1, u_2\} \in \mathcal{U}$ . For  $F \subset E \setminus (E(T) \cup I)$ , the graph  $H = T \cup I \cup F$  contains a  $u_1r$ -path and a  $u_2r$ -path that are internally vertex-disjoint if and only if  $H$  contains 2 internally-disjoint paths to  $r$  from either  $u_1$  or  $u_2$ .*

*Proof:* The proof uses Menger's Theorem and **Property P** of the groups in  $\mathcal{U}$ .

Suppose that  $H$  contains 2 internally vertex-disjoint paths from  $u_1$  to  $r$ . Then  $H$  has no cut-vertex separating  $r$  and  $u_1$ , by Menger's Theorem. In particular, there is no cut-vertex separating  $r$  and  $\{u_1, u_2\}$ . Thus  $H$  contains a  $u_1r$ -path and a  $u_2r$ -path that are internally vertex-disjoint, by Proposition 3.1.

Suppose now that  $H$  contains a  $u_1r$ -path and a  $u_2r$ -path that are internally vertex-disjoint. Now we use **Property P**. If  $u_1 = u_2$ , the proof is complete. Assume therefore that  $u_1 \neq u_2$  and that they have a common parent  $u$  in  $T$  and assume to the contrary that  $H$  has no pair of internally vertex-disjoint  $u_i r$ -paths for  $i = 1, 2$ . Then by Menger's Theorem, there are cut-vertices  $v_1, v_2$  in  $H$ , where  $v_i$  separates  $u_i$  from  $r$ . As  $u_1, u_2$  have a common parent  $u \neq r$ , any vertex separating  $r$  from one of  $u_1, u_2$  must lie on the  $ur$ -path in  $T$ . If  $v_1 = v_2 = v$ , then  $v$  separates both  $u_1, u_2$  from  $r$  contradicting the assumption (by Proposition 3.1). Thus  $v_1 \neq v_2$ , so one of  $v_1, v_2$  is distinct from  $u$ , say  $v_1 \neq u$ . The graph  $H \setminus \{v_1\}$  contains a  $u_2r$ -path. As  $T \setminus \{v_1\}$  contains a  $u_1u_2$ -path, this implies that  $H \setminus \{v_1\}$  contains a  $u_1r$ -path. This contradicts that  $v_1$  separates  $u_1$  and  $r$ . ■

Thus the original density problem can be reduced to the following problem. Given a collection of groups  $\{u_1^i, u_2^i\}$  for  $i = 1, 2, \dots$ , find a subset  $F \subset E \setminus (E(T) \cup I)$  such that the ratio of  $\text{COST}(F)$  to the number of groups  $i$  such that at least one of  $u_1^i$  or  $u_2^i$  has 2-vertex-disjoint paths to  $r$  in  $E(T) \cup I \cup F$ .

The problem of finding a subgraph that minimizes the ratio of its cost over the total profit of vertices that are 2-vertex-connected to the root was studied by Chekuri and Korula [6], who gave an  $O(\log n)$ -approximation for the problem. We use their algorithm to compute an  $O(\log n)$ -approximation for the density version of our problem, as follows. The input to the algorithm of Chekuri and Korula is the original graph with root  $r$  and with *profits* on vertices defined as follows. Let the *profit*  $p(u)$  of a vertex  $u$  be the number of groups  $i$  such that  $u \in \{u_1^i, u_2^i\}$ . Thus  $p(u)$  denotes the number of new groups that would get connected to the root via 2 vertex-disjoint

- (i) Initialize subgraph  $\mathcal{H} \leftarrow \emptyset$  and  $q' \leftarrow q$  to be the number of uncovered groups.
- (ii) If  $q' > 0$ , begin a phase:
  - (a) Assign a cost of zero to all vertices in  $\mathcal{H}$ .
  - (b) **Find Twin-pairs:** Compute a subgraph  $H$  of cost  $O(\text{OPT} \cdot k \log n)$  that contains twin pairs (Definition 3.1) from at least  $q'/2$  uncovered groups.
  - (c) **Cover:** Compute a subgraph  $I$  of cost  $O(\text{OPT} \cdot k \log n)$  that covers at least  $q'/2$  uncovered groups.
  - (d) Update  $\mathcal{H} \leftarrow \mathcal{H} \cup H \cup I$  and update  $q'$  to be the number of uncovered groups in  $\mathcal{H}$ .
- (iii) Output  $\mathcal{H}$ .

Figure 2: An outline of our algorithm for VC-FTGS- $k$  with vertex-costs.

paths provided  $u$  gets connected to the root via 2 vertex-disjoint paths. Note that since both  $u_1^i$  or  $u_2^i$  may claim profit for covering group  $i$ , we may overestimate the profit of 2-vertex connecting a subset of vertices to  $r$  by a factor of 2. This introduces another factor 2 in the approximation. Using Chekuri-Korula algorithm, we compute a subgraph which yields a  $O(\log n)$  approximation for minimizing the ratio of its cost over the total profit of its vertices that are 2-vertex-connected to the root. This subgraph yields  $O(\log n)$  approximation to the problem defined in Lemma 3.1.

Thus the proof is complete.

## 3.2 Algorithm for EC-FTGS- $k$ and VC-FTGS- $k$ with edge/vertex costs

We present an algorithm for VC-FTGS- $k$  with vertex-costs, which is more general than the case of edge-costs. Adaptation of this algorithm to EC-FTGS- $k$  is easy.

Fix an optimum solution OPT with cost also denoted by OPT. To simplify the presentation, the algorithm given below is assumed to know the value of OPT. In reality, the algorithm tries all possible guesses for the power of 2 that is closest to OPT and picks the cheapest solution among those computed for each of these guesses.

Our algorithm has logarithmic number of phases. In each phase, it covers at least half of the remaining groups. A high-level pseudo-code of the algorithm is given in Figure 2. At the beginning of each phase, we make the cost of the vertices that are already picked in the solution  $\mathcal{H}$  zero. In what follows, we analyze a single phase which begins with  $q'$  uncovered groups overall. The groups in  $\mathcal{S}$  that are already covered are ignored. In what follows, we explain how to implement steps **Find Twin-pairs** and **Cover** respectively.

### 3.2.1 How to implement step Find Twin-pairs.

**Definition 3.1** *For a group  $S \in \mathcal{S}$ , we say that the terminals  $s_1, s_2 \in S$  form a twin pair if OPT contains two internally-disjoint paths from  $s_1$  and  $s_2$  to  $r$ . If there are more than one such pairs for a group, we designate exactly one of these pairs as twin pair arbitrarily.*

We do not know which terminals form twin pairs a-priori. Nevertheless, we can compute a low-cost tree that connects the twin pairs from at least half of the remaining groups to the root, as shown below. We iteratively use  $p$ -Steiner tree algorithm for  $p = q'$ . Recall that the  $p$ -Steiner Tree problem is to compute a minimum-cost tree that connects at least  $p$  terminals to the root. Let  $H$  denote the union of the  $p$ -Steiner trees computed so far. Assume that the number of uncovered groups is at least  $q'/2$ .

**Lemma 3.5** *Assign a cost of zero vertices in  $H$ . Now apply  $(c \log n)$ -approximation algorithm [7] for the  $p$ -Steiner tree problem (where  $c$  is a constant) for the instance given by  $H$ , root  $r$ ,  $p = q'$  and terminals as the vertices in the union of uncovered groups in  $\mathcal{S}$  but not in  $H$ :  $\{v \in S \mid S \in \mathcal{S} \text{ is uncovered, } v \notin H\}$ . If the cost of the computed Steiner tree is more than  $(c \log n) \cdot \text{OPT}$ , then  $H$  contains the twin pairs for at least  $q'/2$  uncovered groups.*

*Proof:* Assume on the contrary that  $H$  does not contain twin pairs for at least  $q'/2$  uncovered groups. Thus the OPT solution connects at least  $2 \cdot q'/2 = q'$  terminals in this  $p$ -Steiner tree instance to the root. Since we use a  $(c \log n)$ -approximation, the cost of the computed Steiner tree is at most  $(c \log n) \cdot \text{OPT}$ . This is a contradiction, and the lemma follows. ■

We run the  $p$ -Steiner tree algorithm iteratively while the cost of the new tree computed is at most  $c \log n \cdot \text{OPT}$ . Since  $H$  contains at least  $p = q'$  new terminals in each iteration, the total number of invocations of such  $p$ -Steiner tree algorithm is at most  $2q' \cdot k/q'$ . This holds since the size of each group is at most  $k$ . Thus the total cost of the step **Find Twin-pairs** is  $\text{COST}(H) \leq O(\text{OPT} \cdot k \log n)$ .

### 3.2.2 How to implement step Cover.

Even if  $H$  contains the twin pairs of at least  $q'/2$  uncovered groups, we still do not know which of the terminals in  $H$  form twin pairs. We therefore need one more definition.

**Definition 3.2** *Let  $T$  be a spanning tree of  $H$ . We say that a vertex  $u_1$  can help an uncovered group  $S$  if there exist distinct vertices  $s_1, s_2 \in S \cap T$  and another vertex  $u_2$  so that  $u_1, u_2$  have the same parent  $u = \text{LCA}(s_1, s_2)$  in the tree  $T$ . The profit  $p(u)$  of vertex  $u$  is defined as the number of uncovered groups in  $\mathcal{S}$  that  $u$  can help.*

The intuition of the above definition comes from our algorithm for VC-FTGS-2, and in particular, from Lemma 3.4. Note that the profit of a vertex  $u$  is the number of uncovered groups that will get covered if  $u$  gets connected to the root via 2 internally-disjoint paths.

Since more than one vertex can claim a profit for covering a single group, it is important to understand how many vertices can help a particular group. Since there are at most  $k$  terminals in any group  $S \in \mathcal{S}$ , it is obvious that there can be at most  $O(k^2)$  vertices that can help  $S$ . This crude upper bound comes from the fact that there are  $O(k^2)$  pairs  $s_1, s_2 \in S$  that can give rise to such vertices. However the following lemma presents a careful counting of such vertices.

**Lemma 3.6** *There are  $O(k)$  vertices that can help any single group  $S \in \mathcal{S}$ .*



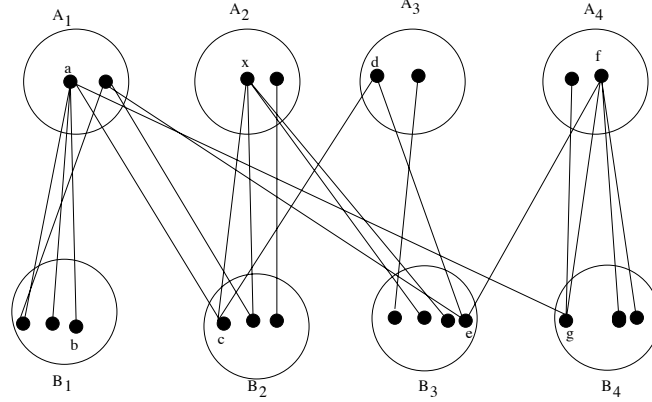


Figure 4: A yes instance. The vertices  $a, x, d, f, b, c, e, g$  are a MIN-REP-cover of the instance

## 4 Proof of Part (iii)

To present our hardness results, we introduce the MIN-REP problem [25]. The input consists of a bipartite graph  $G = (V_1, V_2, E)$ , and two disjoint partitions  $V_1 = \bigcup_{i=1}^q A_i$  and  $V_2 = \bigcup_{i=1}^q B_i$  of  $V_1$  and  $V_2$  respectively. Let  $n = |V_1| + |V_2|$ . The above instance induces a bipartite *supergraph*  $\tilde{G} = (\{A_i\}, \{B_i\}, \bar{E})$ . The *supervertices* of this supergraph are  $\{A_i\}_{i=1}^q \cup \{B_i\}_{i=1}^q$ . A *superedge* between  $A$  and  $B$  is added if there exist  $a \in A, b \in B$  so that  $(a, b) \in E$ . We say that two vertices  $a \in A$  and  $b \in B$  *Min-Rep-cover*  $(A, B)$  if  $(a, b) \in E$ . A subset  $C \subseteq V_1 \cup V_2$  of vertices is said to *cover* a superedge  $(A, B)$  if it contains a pair  $a \in A$  and  $b \in B$  that covers  $(A, B)$ . A subset  $C$  is a *Min-Rep-cover* if it covers all the superedges.

**Theorem 4.1 (Raz [25])** *For any constant  $\epsilon > 0$ , there is a reduction from any NP-complete problem instance to a MIN-REP instance  $(G, \tilde{G})$  so that for a yes instance, there exists a set  $C$  containing one vertex out of any supervertex that is a MIN-REP-cover, but for a no instance, every MIN-REP-cover contains at least  $q \cdot 2^{\log^{1-\epsilon} n}$  vertices of  $V_1 \cup V_2$ .*

We reduce MIN-REP instance to an instance  $\hat{G}$  of directed **Single EC-FTGS**. Add all the vertices of the MIN-REP instance to  $\hat{G}$ . For every edge in  $e = (a, b) \in E$ , add a vertex  $v_e$  to  $\hat{G}$  in the middle of  $e$  and replace  $e$  by two directed edges from  $v_e$  to each of  $a$  and  $b$ . These edges have cost 0. Add a root  $r$  and add a directed edge of cost 1 from every vertex in  $V_1 \cup V_2$  to  $r$ . Every group corresponds to a superedge  $(A, B)$ . Let  $g_{AB}$  be this group. For every  $e = (a, b)$  where  $a \in A, b \in B$ , add  $v_e$  to  $g_{AB}$ . This finishes the description of the reduction.

We show that every solution to the **Single EC-FTGS** instance  $\hat{G}$  defines a solution for the MIN-REP instance of the same cost and a solution for MIN-REP defines a solution for **Simple EC-FTGS** instance  $\hat{G}$  of the same cost.

First consider a solution for the **Single EC-FTGS** instance  $\hat{G}$ . Consider a superedge  $(A, B)$ . The only way to have two edge-disjoint paths from a single vertex in  $g_{AB}$  to  $r$  is for some  $v_e, e = (a, b)$  to have two edge-disjoint paths to  $r$ :  $v_e \mapsto a \mapsto r$  and  $v_e \mapsto b \mapsto r$ . This incurs a cost 1 for the edge  $ar$  and a cost of 1 for the edge  $br$ . From the construction, the collection of such end-points  $\{a, b\}$  together induces a MIN-REP cover of cost at most the cost of the **Single EC-FTGS** solution.

Conversely, given a MIN-REP solution, every  $e = (a, b)$  covering the superedge  $(A, B)$  gives in an obvious way two disjoint paths to  $r$  from  $v_e$ , adding 2 to the cost of the Single EC-FTGS solution. Thus cost of the solution for Single EC-FTGS is at most the cost (number number of of vertices of  $V_1 \cup V_2$  selected) in the solution for the MIN-REP instance [12]. This completes the proof of the hardness of approximation part of Part (iii) in Theorem 1.1.

*Remark:* We note that a similar hardness results hold for the vertex-connectivity version or vertex-costs.

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