CS 171
Lecture Outline
February 17, 2010

Example 1. Given a sequence of n integers, show that there exists a subsequence of consecutive integers whose sum is a multiple of n.

Solution. Let $x_1, x_2, \ldots, x_n$ be the sequence of n integers. Consider the following n sums.

$$x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \cdots + x_n$$

If any of these n sums is divisible by n, then we are done. Otherwise, each of the n sums have a non-zero remainder when divided by n. There are at most $n - 1$ different possible remainders: $1, 2, \ldots n - 1$. Since there are n sums, by the pigeonhole principle, at least two of the n sums have the same remainder when divided by n. Let p and q, $p < q$, be integers such that for some integers c_1 and c_2,

$$x_1 + x_2 + \cdots + x_p = c_1 n + r \text{ and } x_1 + x_2 + \cdots + x_q = c_2 n + r$$

Subtracting the two sums, we get

$$x_{p+1} + \cdots + x_q = (c_2 - c_1)n$$

Hence, $x_{p+1} + \cdots + x_q$ is divisible by n.

Example 2. Show that in any group of six people there are either three mutual friends or three mutual strangers.

Solution. Consider one of the six people, say A. The remaining five people are either friends of A or they do not know A. By the pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of the five people are either friends of A or are unacquainted with A. In the former case, if any two of the three people are friends then these two along with A would be mutual friends, otherwise the three people would be strangers to each other. The proof for the latter case, when three or more people are unacquainted with A, proceeds in the same manner.

Example 3. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists consecutive days during which the chess master will have played exactly 21 games.

Solution. Let a_i, $1 \le i \le 77$, be the total number of games that the chess master has played during the first i days. Note that the sequence of numbers $a_1, a_2, \ldots, a_{77}$ is a strictly increasing sequence. We have

$$1 \le a_1 < a_2 < \ldots < a_{77} \le 11 \times 12 = 132$$

Now consider the sequence $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$. We have

$$22 \le a_1 + 21 < a_2 + 21 < \ldots < a_{77} + 21 \le 153$$

Clearly, this sequence is also a strictly increasing sequence. The numbers $a_1, a_2, \ldots, a_{77}, a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ (154 in all) belong to the set $\{1, 2, \ldots, 153\}$. By the pigeonhole principle there must be two numbers out of the 154 numbers that must be the same. Since no two numbers in $a_1, a_2, \ldots, a_{77}$ are equal and no two numbers in $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are equal there must exist i and j such that $a_i = a_j + 21$. Hence during the days $j + 1, j + 2, \ldots, i$, exactly 21 games must have been played.