H. W. for CH 7

Please study K-4, K-9, K-10
P-3, P-6.

(a) Draw the pole P(m).
Draw a line thru P(m) & l to l.

(b) Since the angle α between l & m can be made arbitrarily small, the angle of parallelism can be any acute angle.

(c) Line l, m, n has no common transversal.
Assume \( P(e) \) lies on the line extending \( m \).

We now prove that \( P(m) \) lies on the line extending \( l \).

We give an analytical proof.

Suppose \( T \) is a circle with radius \( R \) and center at the origin. Assume the \( P(e) = (\alpha, 0) \). Since \( O \overrightarrow{P(e)} \cdot \overrightarrow{OE} = R^2 \),

Thus \( E = (\frac{R^2}{\alpha}, 0) \).

Let \( F = (\beta_1, \beta_2) \). Since \( \overrightarrow{OF} \perp \overrightarrow{P(e)} \), we have

\[
\frac{\beta_2}{\beta_1} \cdot \frac{\beta_2 - 0}{\beta_1 - \alpha} = -1 \Rightarrow \beta_2 = -\beta_1(\beta_1 - \alpha)
\]

Hence \( \beta_1^2 + \beta_2^2 = \lambda \beta_1 \).

The equation for the line \( \overrightarrow{OF} \) is \( y = \frac{\beta_2}{\beta_1} x \), which meets the vertical line \( \overrightarrow{AB} : x = \frac{R^2}{\alpha} \) at the point \( Q = (\frac{R^2}{\alpha}, \frac{\beta_2 R^2}{\beta_1 \alpha}) \).

Since \( \overrightarrow{OQ} = \sqrt{\left(\frac{R^2}{\alpha}\right)^2 + \left(\frac{\beta_2 R^2}{\beta_1 \alpha}\right)^2} = \frac{R^2}{\alpha} \sqrt{1 + \frac{\beta_2^2}{\beta_1^2}} = \frac{R^2}{\alpha} \sqrt{\frac{\alpha \beta_1}{\beta_1^2}} = \frac{R^2}{\alpha} \sqrt{\frac{\alpha \beta_1}{\beta_1^2}} = \frac{R^2}{\alpha} \frac{\alpha \beta_1}{\beta_1^2} = \frac{R^2}{\alpha^2} \sqrt{\frac{\alpha \beta_1}{\beta_1^2}}\) and \( \overrightarrow{OF} = \sqrt{\beta_1^2 + \beta_2^2} = \sqrt{\lambda \beta_1} \), we have

\( \overrightarrow{OQ} \cdot \overrightarrow{OF} = R^2 \) Hence \( Q \) is the pole of \( CD \).

Thus \( \overrightarrow{AB} \) passes through the pole of \( m \).
Let \( z = x_1 + ix_2 \), which has coordinates \((x_1, x_2, 0)\).

The line \( \overrightarrow{Nz} \) has parametric equation
\[
\begin{align*}
x_1 &= o + tx_1 \\
x_2 &= o + tx_2 \\
x_3 &= 1 - t.
\end{align*}
\]

We now find the intersection \( P \) of this line with the unit sphere:
\[
(tx_1)^2 + (tx_2)^2 + (1-t)^2 = 1
\]
\[
\Rightarrow t^2(1+x_1^2+x_2^2) - 2t = 0
\]
\[
\Rightarrow t = 0 \text{ or } t = \frac{2}{1+x_1^2+x_2^2}.
\]

Hence \( P = \left( \frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2}, 1 - \frac{2}{1+x_1^2+x_2^2} \right) \).

Hence \( F(z) = \frac{2x_1}{1+x_1^2+x_2^2} + i \frac{2x_2}{1+x_1^2+x_2^2} = \frac{2(x_1+i x_2)}{1+x_1^2+x_2^2} \)
\[
= \frac{2z}{1+|z|^2}.
\]

Let \( W = F(z) = (u_1, u_2) \). Then \( P = (u_1, u_2, -\sqrt{1-u_1^2-u_2^2}) \).

Hence the line \( \overrightarrow{NP} \) is given by:
\[
\begin{align*}
x_1 &= tu_1, \\
x_2 &= tu_2, \\
x_3 &= 1 + t(-\sqrt{1-u_1^2-u_2^2} - 1).
\end{align*}
\]

Letting \( z = 0 \):
\[
1 - t(\sqrt{1-u_1^2-u_2^2} + 1) = 0 \Rightarrow t = \frac{1}{1+\sqrt{1-1|w|^2}}
\]

Thus \( z = F^{-1}(w) = \frac{u_1 + i u_2}{1+\sqrt{1-1|w|^2}} = \frac{w}{1+\sqrt{1-1|w|^2}} \) which is the

Formula for the inverse.
(a) \[ d'(AB) = \frac{1}{2} | \ln (AB, PQ) | \]

\[ = \frac{1}{2} | \ln \left( \frac{AP}{A^Q} \cdot \frac{BQ}{B^P} \right) | = \frac{1}{2} | \ln \left( \frac{1}{\sqrt{3}} \right) | \]

\[ = \frac{1}{2} \ln 3. \]

(b) Suppose \( M = (0, y) \) is the midpoint of \( AB \), then

\[ d'(MA) = d'(MB) = \frac{1}{2} d'(AB) \]

\[ \text{Hence} \]

\[ \frac{1}{2} | \ln (MA, PQ) | = \frac{1}{4} \ln 3 \]

\[ = \frac{1}{2} | \ln \left( \frac{MP}{MQ} \cdot \frac{AQ}{AP} \right) | = \frac{1}{2} \ln 3 \]

\[ \ln \left( \frac{y+1}{1-y} \right) = \ln \sqrt{3} \quad \Rightarrow \quad \frac{y+1}{1-y} = \sqrt{3} \]

\[ \Rightarrow \quad y+1 = \sqrt{3} (1-y), \quad y+1 = \sqrt{3} - \sqrt{3} y \quad \Rightarrow \quad y = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \]

\[ \text{Hence} M = (0, \frac{\sqrt{3} - 1}{\sqrt{3} + 1}) = (0, \frac{\sqrt{3} - 1}{3-1}) = (0, 2-\sqrt{3}). \]

(C) Let \( X = (x, y) \) be a point that are equal distance to \( l \) as \( B \). Let \( Y = (x, 0) \). Then

\[ d'(XY) = d'(AB) = \frac{1}{2} \ln 3. \]

\[ \text{Hence} \]

\[ \frac{1}{2} | \ln \left( \frac{XP'}{XQ'} \cdot \frac{YP'}{YP} \right) | = \frac{1}{2} \ln \left( \frac{\sqrt{1-x^2}+y}{1-x^2-y} \right) \cdot \frac{\sqrt{1-x^2}}{1-x^2-y} = \frac{1}{2} \ln 3 \]

\[ \Rightarrow \quad \frac{\sqrt{1-x^2}+y}{1-x^2-y} = 3 \quad \Rightarrow \quad \sqrt{1-x^2}+y = 3(1-x^2)-3y \]

\[ \Rightarrow \quad 2y = \sqrt{1-x^2} \quad \Rightarrow \quad 4y^2 + x^2 = 1, \text{ which is an equation for an ellipse}. \]
Given: \( A' \) is the inverse of \( A \) in \( \delta \); \( \alpha \) not thru \( C \);
\( A' \) is the inverse of \( A \) in \( \delta \);
Show: \( A' \) is the inverse of \( C \) in \( \alpha' \).

Proof:
By Prop 7.5, \( A' \) is the inverse of \( C \) in \( \alpha' \) if and only if any circle thru \( A' \) & \( C \) is \( \perp \) to \( \alpha' \).

Now let \( \beta \) be any circle thru \( C \) & \( A' \). Then the inverse of \( \beta \) in \( \delta \) is a line thru \( A \). Since \( A \) is the center of \( \alpha \), we have \( \beta' \perp \alpha' \). Since angle is preserved under inversion, \( \beta' \perp \alpha' \).
Step 1: Pick a point B on P-ray $\overrightarrow{AP}$. Draw a tangent $t$ to the arc $\widehat{AP}$. Let $t$ meet $\overrightarrow{OA}$ at $C$. Draw a circle with center $C$ thru $B$. This circle meets the other side $\overrightarrow{AQ}$ at $B'$. By the construction, the Poincaré length $d(AB) = d(AB')$. The $P$-angle-bisector of angle $\angleBAB'$ is then the $P$-perpendicular bisector of $P$-segment $BB'$. In Step 2, we will show how to construct the $I$-bisector.

Step 2: We now show how to construct $P$-perpendicular bisector of $P$-segment $BB'$. (See next page for the diagram.)

(1) Let $Q$ be the center of “$P$-line thru $B$&$B'$”

(2) The ray $\overrightarrow{QB}$ meet $t'$ at $A$&$C$; the ray $\overrightarrow{QB'}$ meet $t'$ at $A'$&$C'$.

(3) $\overrightarrow{AA'}$ & $\overrightarrow{CC'}$ meet at a point $P$.

(4) From $P$, draw a tangent to $t'$, tangent at $T$.

(5) The circle with center $P$ and thru $T$ is the desired $P$-perpendicular bisector of $P$-segment $BB'$.
(See diagram)

\( \overline{OQ} = \sqrt{2} \). Hence

\( \overline{OP} = \sqrt{2} - 1 \)

By Lemma 7.4

\[ \frac{e^d - 1}{e^d + 1} = (\sqrt{2} - 1) \]

\[ \Rightarrow e^d - 1 = (\sqrt{2} - 1)(e^d + 1) \]

\[ \Rightarrow e^d = \frac{\sqrt{2} - 1 + 1}{1 - (\sqrt{2} - 1)} = \frac{\sqrt{2}}{2 - \sqrt{2}} = \frac{\sqrt{2}(2 + \sqrt{2})}{4 - 2} = \sqrt{2} + 1 \]

Hence \( d = \log(\sqrt{2} + 1) \).

\( \Box \)