POSITIVITY OF THE $\bar{\partial}$-NEUMANN LAPLACIAN

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Dedicated to Professor Linda Rothschild

Abstract. We study the $\bar{\partial}$-Neumann Laplacian from spectral theoretic perspectives. In particular, we show how pseudoconvexity of a bounded domain is characterized by positivity of the $\bar{\partial}$-Neumann Laplacian.

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1. Introduction

Whether or not a given system has positive ground state energy is a widely studied problem with significant repercussions in physics, particularly in quantum mechanics. It follows from the classical Hardy inequality that the bottom of the spectrum of the Dirichlet Laplacian on a domain in $\mathbb{R}^n$ that satisfies the outer cone condition is positive if and only if its inradius is finite (see [D95]). Whereas spectral behavior of the Dirichlet Laplacian is insensitive to boundary geometry, the story for the $\bar{\partial}$-Neumann Laplacian is different. Since the work of Kohn [Ko63, Ko64] and Hörmander [H65], it has been known that existence and regularity of the $\bar{\partial}$-Neumann Laplacian closely depend on the underlying geometry (see the surveys [BSt99, Ch99, DK99, FS01] and the monographs [CS99, St09]).

Let $\Omega$ be a domain in $\mathbb{C}^n$. It follows from the classical Theorem B of Cartan that if $\Omega$ is pseudoconvex, then the Dolbeault cohomology groups $H^{0,q}(\Omega)$ vanish for all $q \geq 1$. (More generally, for any coherent analytic sheaf $\mathcal{F}$ over a Stein manifold, the sheaf cohomology groups $H^q(X,\mathcal{F})$ vanish for all $q \geq 1$.) The converse is also true ([Se53], p. 65). Cartan’s Theorem B and its converse were generalized by Laufer [L66] and Siu [Siu67] to a Riemann domain over a Stein manifold. When $\Omega$ is bounded, it follows from Hörmander’s $L^2$-existence theorem for the $\bar{\partial}$-operator that if $\Omega$ is in addition pseudoconvex, then the $L^2$-cohomology groups $\tilde{H}^{0,q}(\Omega)$ vanish for $q \geq 1$. The converse of Hörmander’s theorem also holds, under the assumption that the interior of the closure of $\Omega$ is the domain itself. Sheaf theoretic arguments for the Dolbeault cohomology groups can be modified to give a proof of this fact (cf. [Se53, L66, Siu67, Br83, O88]; see also [Fu05] and Section 3 below).

In this expository paper, we study positivity of the $\bar{\partial}$-Neumann Laplacian, in connection with the above-mentioned classical results, through the lens of spectral theory. Our emphasis is on the interplay between spectral behavior of the $\bar{\partial}$-Neumann Laplacian and the geometry of the domains. This is evidently motivated by Marc Kac’s famous question “Can one hear the shape of a drum?” [Ka66]. Here we are interested in determining the geometry of a domain in $\mathbb{C}^n$ from the spectrum of the $\bar{\partial}$-Neumann Laplacian. (See [Fu05, Fu08] for related results.) We make an effort to present a more accessible and self-contained
treatment, using extensively spectral theoretic language but bypassing sheaf cohomology theory.

2. Preliminaries

In this section, we review the spectral theoretic setup for the \(\overline{\partial}\)-Neumann Laplacian. The emphasis here is slightly different from the one in the extant literature (cf. [FK72, CS99]). The \(\overline{\partial}\)-Neumann Laplacian is defined through its associated quadratic form. As such, the self-adjoint property and the domain of its square root come out directly from the definition.

Let \(Q\) be a non-negative, densely defined, and closed sesquilinear form on a complex Hilbert space \(H\) with domain \(D(Q)\). Then \(Q\) uniquely determines a non-negative and self-adjoint operator \(S\) such that \(D(S^{1/2}) = D(Q)\) and

\[
Q(u, v) = \langle S^{1/2}u, S^{1/2}v \rangle
\]

for all \(u, v \in D(Q)\). (See Theorem 4.4.2 in [D95], to which we refer the reader for the necessary spectral theoretic background used in this paper.) For any subspace \(L \subset D(Q)\), let \(\lambda(L) = \sup\{Q(u, u) \mid u \in L, \|u\| = 1\}\). For any positive integer \(j\), let

\[
(2.1) \quad \lambda_j(Q) = \inf\{\lambda(L) \mid L \subset D(Q), \dim(L) = j\}.
\]

The resolvent set \(\rho(S)\) of \(S\) consists of all \(\lambda \in \mathbb{C}\) such that the operator \(S - \lambda I: D(S) \to H\) is both one-to-one and onto (and hence has a bounded inverse by the closed graph theorem).

The spectrum \(\sigma(S)\), the complement of \(\rho(S)\) in \(\mathbb{C}\), is a non-empty closed subset of \([0, \infty)\). Its bottom \(\inf \sigma(S)\) is given by \(\lambda_1(Q)\). The essential spectrum \(\sigma_e(S)\) is a closed subset of \(\sigma(S)\) that consists of isolated eigenvalues of infinite multiplicity and accumulation points of the spectrum. It is empty if and only if \(\lambda_j(Q) \to \infty\) as \(j \to \infty\). In this case, \(\lambda_j(Q)\) is the \(j\)th eigenvalue of \(S\), arranged in increasing order and repeated according to multiplicity.

The bottom of the essential spectrum \(\inf \sigma_e(T)\) is the limit of \(\lambda_j(Q)\) as \(j \to \infty\). (When \(\sigma_e(S) = \emptyset\), we set \(\inf \sigma_e(S) = \infty\).)

Let \(T_k: H_k \to H_{k+1}\), \(k = 1, 2\), be densely defined and closed operators on Hilbert spaces. Assume that \(\mathcal{R}(T_1) \subset \mathcal{N}(T_2)\), where \(\mathcal{R}\) and \(\mathcal{N}\) denote the range and kernel of the operators. Let \(T_k^*\) be the Hilbert space adjoint of \(T_k\), defined in the sense of Von Neumann by

\[
\mathcal{D}(T_k^*) = \{u \in H_{k+1} \mid \exists C > 0, |\langle u, T_k v \rangle| \leq C \|v\|, \forall v \in \mathcal{D}(T_k)\}
\]

and

\[
\langle T_k^* u, v \rangle = \langle u, T_k v \rangle, \quad \text{for all } u \in \mathcal{D}(T_k^*) \text{ and } v \in \mathcal{D}(T_k).
\]

Then \(T_k^*\) is also densely defined and closed. Let

\[
Q(u, v) = \langle T_k^* u, T_k^* v \rangle + \langle T_2 u, T_2 v \rangle
\]

with its domain given by \(\mathcal{D}(Q) = \mathcal{D}(T_k^*) \cap \mathcal{D}(T_2)\). The following proposition elucidates the above approach to the \(\overline{\partial}\)-Neumann Laplacian.

**Proposition 2.1.** \(Q(u, v)\) is a densely defined, closed, non-negative sesquilinear form. The associated self-adjoint operator \(\Box\) is given by

\[
(2.2) \quad \mathcal{D}(\Box) = \{f \in H_2 \mid f \in \mathcal{D}(Q), T_2 f \in \mathcal{D}(T_2^*), T_1^* f \in \mathcal{D}(T_1)\}, \quad \Box = T_1 T_1^* + T_2 T_2.
\]

**Proof.** The closedness of \(Q\) follows easily from that of \(T_1\) and \(T_2\). The non-negativity is evident. We now prove that \(\mathcal{D}(Q)\) is dense in \(H_2\). Since \(\mathcal{N}(T_2)^\perp = \overline{\mathcal{R}(T_2^*)} \subset \mathcal{N}(T_1^*)\) and

\[
\mathcal{D}(T_2) = \mathcal{N}(T_2) \oplus (\mathcal{D}(T_2) \cap \mathcal{N}(T_2)^\perp),
\]
we have
\[ \mathcal{D}(Q) = \mathcal{D}(T_1^*) \cap \mathcal{D}(T_2) = (\mathcal{N}(T_2) \cap \mathcal{D}(T_1^*)) \oplus (\mathcal{D}(T_2) \cap \mathcal{N}(T_2)^\perp). \]

Since \( \mathcal{D}(T_1^*) \) and \( \mathcal{D}(T_2) \) are dense in \( H_2 \), \( \mathcal{D}(Q) \) is dense in \( \mathcal{N}(T_2) \oplus \mathcal{N}(T_2)^\perp = H_2 \).

It follows from the above definition of \( \square \) that if \( f \in \mathcal{D}(\square) \) if and only if \( f \in \mathcal{D}(Q) \) and there exists a \( g \in H_2 \) such that
\[
Q(u, f) = \langle u, g \rangle, \quad \text{for all } u \in \mathcal{D}(Q)
\]
(cf. Lemma 4.4.1 in [D95]). Thus
\[ \mathcal{D}(\square) \supset \{ f \in H_2 \mid f \in \mathcal{D}(Q), T_2 f \in \mathcal{D}(T_1^*), T_1^* f \in \mathcal{D}(T_1) \}. \]

We now prove the opposite containment. Suppose \( f \in \mathcal{D}(\square) \). For any \( u \in \mathcal{D}(T_2) \), we write \( u = u_1 + u_2 \in (\mathcal{N}(T_1^*) \cap \mathcal{D}(T_2)) \oplus \mathcal{N}(T_1^*)^\perp \). Note that \( \mathcal{N}(T_1^*)^\perp \subset \mathcal{R}(T_1^*) = \mathcal{N}(T_1) \). It follows from (2.3) that
\[
|\langle T_1^* u, f \rangle| = |\langle T_2 u_1, T_2 f \rangle| = |Q(u_1, f)| = |\langle u_1, g \rangle| \leq ||u_1|| \cdot ||g||.
\]
Hence \( T_2 f \in \mathcal{D}(T_1^*) \). The proof of \( T_1^* f \in \mathcal{D}(T_1) \) is similar. For any \( w \in \mathcal{D}(T_1^*) \), we write \( w = w_1 + w_2 \in (\mathcal{N}(T_2) \cap \mathcal{D}(T_1^*)) \oplus \mathcal{N}(T_2)^\perp \). Note that \( \mathcal{N}(T_2)^\perp = \mathcal{R}(T_2) \subset \mathcal{N}(T_1^*) \).

Therefore, by (2.3),
\[
|\langle T_1^* w, T_1^* f \rangle| = |\langle T_2^* w_1, T_2^* f \rangle| = |Q(w_1, f)| = |\langle w_1, g \rangle| \leq ||w_1|| \cdot ||g||.
\]
Hence \( T_1^* f \in \mathcal{D}(T_1^*) = \mathcal{D}(T_1) \). It follows from the definition of \( \square \) that for any \( f \in \mathcal{D}(\square) \) and \( u \in \mathcal{D}(Q) \),
\[
\langle \square f, u \rangle = \langle \square^{1/2} f, \square^{1/2} u \rangle = Q(f, u) = \langle T_1^* f, T_1^* u \rangle + \langle T_2 f, T_2 u \rangle = \langle (T_1^* + T_2^*) f, u \rangle.
\]
Hence \( \square = T_1 T_1^* + T_2 T_2^* \).

The following proposition is well-known (compare [H65], Theorem 1.1.2 and Theorem 1.1.4; [C83], Proposition 3; and [Sh92], Proposition 2.3). We provide a proof here for completeness.

**Proposition 2.2.** \( \inf \sigma(\square) > 0 \) if and only if \( \mathcal{R}(T_1) = \mathcal{N}(T_2) \) and \( \mathcal{R}(T_2) \) is closed.

**Proof.** Assume \( \inf \sigma(\square) > 0 \). Then 0 is in the resolvent set of \( \square \) and hence \( \square \) has a bounded inverse \( G : H_2 \to \mathcal{D}(\square) \). For any \( u \in H_2 \), write \( u = T_1 T_1^* G u + T_2 T_2^* G u \). If \( u \in \mathcal{N}(T_1) \), then \( 0 = (T_2 u, T_2 G u) = (T_2^* T_2 G u, T_2 G u) = (T_2 T_2^* G u, T_2 G u) \). Hence \( T_2^* T_2 G u = 0 \) and \( u = T_1 T_1^* G u \). Similarly, \( \mathcal{R}(T_1) = \mathcal{N}(T_2) \). Therefore, \( \mathcal{R}(T_1) = \mathcal{N}(T_2) \).

Therefore \( T_2^* \) and hence \( T_2 \) have closed range. To prove the opposite implication, we write \( u = u_1 + u_2 \in \mathcal{N}(T_2) \oplus \mathcal{N}(T_2)^\perp \), for any \( u \in \mathcal{D}(Q) \). Note that \( u_1, u_2 \in \mathcal{D}(Q) \). It follows from \( \mathcal{N}(T_2) = \mathcal{R}(T_1) \) and the closed range property of \( T_2 \) that there exists a positive constant \( c \) such that \( c ||u_1||^2 \leq ||T_1^* u_1||^2 \) and \( c ||u_2||^2 \leq ||T_2 u_2||^2 \). Thus
\[
c ||u||^2 = c (||u_1||^2 + ||u_2||^2) \leq ||T_1^* u_1||^2 + ||T_2 u_2||^2 = Q(u, u).
\]
Hence \( \inf \sigma(\square) \geq c > 0 \) (cf. Theorem 4.3.1 in [D95]).

Let \( \mathcal{N}(Q) = \mathcal{N}(T_1^*) \cap \mathcal{N}(T_2) \). Note that when it is non-trivial, \( \mathcal{N}(Q) \) is the eigenspace of the zero eigenvalue of \( \square \). When \( \mathcal{R}(T_1) \) is closed, \( \mathcal{N}(T_2) = \mathcal{R}(T_1) \oplus \mathcal{N}(Q) \). For a subspace \( L \subset H_2 \), denote by \( P_L \perp \) the orthogonal projection onto \( L^\perp \) and \( T_2 |_{L^\perp} \) the restriction of \( T_2 \) to \( L^\perp \). The next proposition clarifies and strengthens the second part of Lemma 2.1 in [Fut05].

**Proposition 2.3.** The following statements are equivalent:
(1) \( \inf \sigma_e(\square) > 0 \).
(2) \( \mathcal{R}(T_1) \) and \( \mathcal{R}(T_2) \) are closed and \( \mathcal{N}(Q) \) is finite dimensional.
(3) There exists a finite dimensional subspace \( L \subset \mathcal{D}(T_1^*) \cap \mathcal{N}(T_2) \) such that \( \mathcal{N}(T_2) \cap L\perp = P_{L\perp} (\mathcal{R}(T_1)) \) and \( \mathcal{R}(T_2|_{L\perp}) \) is closed.

**Proof.** We first prove (1) implies (2). Suppose \( a = \inf \sigma_e(\square) > 0 \). If \( \inf \sigma(\square) > 0 \), then \( \mathcal{N}(Q) \) is trivial and (2) follows from Proposition 2.2. Suppose \( \inf \sigma(\square) = 0 \). Then \( \sigma(\square) \cap [0,a) \) consists only of isolated points, all of which are eigenvalues of finite multiplicity of \( \square \) (cf. Theorem 4.5.2 in [D95]). Hence \( \mathcal{N}(Q) \), the eigenspace of the eigenvalue 0, is finite dimensional. Choose a sufficiently small \( c > 0 \) so that \( \sigma(\square) \cap [0,c) = \{0\} \). By the spectral theorem for self-adjoint operators (cf. Theorem 2.5.1 in [D95]), there exists a finite regular Borel measure \( \mu \) on \( \sigma(\square) \times \mathbb{N} \) and a unitary transformation \( U : H_2 \rightarrow L^2(\sigma(\square) \times \mathbb{N}, d\mu) \) such that \( U \mathcal{N}(Q)^{\perp} = M_\mu \), where \( M_\mu \varphi(x,n) = \chi(x,n) \) is the multiplication operator by \( x \) on \( L^2(\sigma(\square) \times \mathbb{N}, d\mu) \). Let \( P_{\mathcal{N}(Q)} \) be the orthogonal projection onto \( \mathcal{N}(Q) \). For any \( f \in \mathcal{D}(Q) \cap \mathcal{N}(Q)^{\perp} \),

\[
UP_{\mathcal{N}(Q)}f = \chi_{[0,c)\cap [0,c]}Uf = 0
\]

where \( \chi_{[0,c)} \) is the characteristic function of \( [0,c) \). Hence \( Uf \) is supported on \( [c, \infty) \). Therefore,

\[
Q(f,f) = \int_{\sigma(\square) \times \mathbb{N}} x |Uf|^2 d\mu \geq c \|Uf\|^2 = c \|f\|^2.
\]

It then follows from Theorem 1.1.2 in [H65] that both \( T_1 \) and \( T_2 \) have closed range.

To prove (2) implies (1), we use Theorem 1.1.2 in [H65] in the opposite direction: There exists a positive constant \( c \) such that

\[
c \|f\|^2 \leq Q(f,f), \quad \text{for all } f \in \mathcal{D}(Q) \cap \mathcal{N}(Q)^{\perp}.
\]

Proving by contradiction, we assume \( \inf \sigma_e(\square) = 0 \). Let \( \varepsilon \) be any positive number less than \( c \). Since \( L_{[0,\varepsilon)} = \mathcal{R}(\chi_{[0,\varepsilon)}(\square)) \) is infinite dimensional (cf. Lemma 4.1.4 in [D95]), there exists a non-zero \( g \in L_{[0,\varepsilon)} \) such that \( g \perp \mathcal{N}(Q) \). However,

\[
Q(g,g) = \int_{\sigma(\square) \times \mathbb{N}} x \chi_{[0,\varepsilon)}(x) |Ug|^2 d\mu \leq \varepsilon \|Ug\|^2 = \varepsilon \|g\|^2,
\]

contradicting (2.4).

We do some preparations before proving the equivalence of (3) with (1) and (2). Let \( L \) be any finite dimensional subspace of \( \mathcal{D}(T_1^*) \cap \mathcal{N}(T_2) \). Let \( H_2' = H_2 \cap L \). Let \( T_2' = T_2|_{H_2'} \) and let \( T_1' = T_1|_{H_2} \). Then \( T_2' : H_2' \rightarrow H_3 \) and \( T_1' : H_2' \rightarrow H_1 \) are densely defined, closed operators. Let \( T_1' : H_1 \rightarrow H_2' \) be the adjoint of \( T_1' \). It follows from the definitions that \( \mathcal{D}(T_1) \subset \mathcal{D}(T_1') \). The finite dimensionality of \( L \) implies the opposite containment. Thus, \( \mathcal{D}(T_1) = \mathcal{D}(T_1') \). For any \( f \in \mathcal{D}(T_1) \) and \( g \in \mathcal{D}(T_1') \),

\[
\langle T_1' f, g \rangle = \langle f, T_1' g \rangle = \langle f, T_1^* g \rangle = \langle T_1 f, g \rangle.
\]

Hence \( T_1' = P_{L\perp} \circ T_1 \) and \( \mathcal{R}(T_1') = P_{L\perp}(\mathcal{R}(T_1)) \subset \mathcal{N}(T_2') \). Let

\[
Q'(f,g) = \langle T_1'^* f, T_1'^* g \rangle = \langle T_1 f, g \rangle + \langle T_2 f, T_2 g \rangle
\]

be the associated sesquilinear form on \( H_2' \) with \( \mathcal{D}(Q') = \mathcal{D}(Q) \cap L\perp \).

We are now in position to prove that (2) implies (3). In this case, we can take \( L = \mathcal{N}(Q) \). By Theorem 1.1.2 in [H65], there exists a positive constant \( c \) such that

\[
Q(f,f) = Q'(f,f) \geq c \|f\|^2, \quad \text{for all } f \in \mathcal{D}(Q').
\]
We then obtain (3) by applying Proposition 2.2 to $T_1'$, $T_2'$, and $Q'(f, g)$.

Finally, we prove (3) implies (1). Applying Proposition 2.2 in the opposite direction, we know that there exists a positive constant $c$ such that

$$Q(f, f) \geq c\|f\|^2, \quad \text{for all } f \in \mathcal{D}(Q) \cap L^\perp.$$  

The rest of the proof follows the same lines of the above proof of the implication (2) $\Rightarrow$ (1), with $\mathcal{N}(Q)$ there replaced by $L$.

We now recall the definition of the $\overline{\partial}$-Neumann Laplacian on a complex manifold. Let $X$ be a complex hermitian manifold of dimension $n$. Let $C^\infty_{(0,q)}(X) = C^\infty(X, \Lambda^{0,q} T^* X)$ be the space of smooth $(0,q)$-forms on $X$. Let $\overline{\partial}_q : C^\infty_{(0,q)}(X) \to C^\infty_{(0,q+1)}(X)$ be the composition of the exterior differential operator and the projection onto $C^\infty_{(0,q+1)}(X)$.

Let $\Omega$ be a domain in $X$. For $u, v \in C^\infty_{(0,q)}(X)$, let $\langle u, v \rangle$ be the point-wise inner product of $u$ and $v$, and let

$$\langle \langle u, v \rangle \rangle = \int_{\Omega} \langle u, v \rangle dV$$

be the inner product of $u$ and $v$ over $\Omega$. Let $L^2_{(0,q)}(\Omega)$ be the completion of the space of compactly supported forms in $C^\infty_{(0,q)}(\Omega)$ with respect to the above inner product. The operator $\overline{\partial}_q$ has a closed extension on $L^2_{(0,q)}(\Omega)$. We also denote the closure by $\overline{\partial}_q$. Thus $\overline{\partial}_q : L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ is densely defined and closed. Let $\overline{\partial}_q^*$ be its adjoint. For $1 \leq q \leq n-1$, let

$$Q_q(u, v) = \langle \overline{\partial}_q u, \overline{\partial}_q^* v \rangle \rangle + \langle \overline{\partial}_{q-1} u, \overline{\partial}_{q-1}^* v \rangle \rangle$$

be the sesquilinear form on $L^2_{(0,q)}(\Omega)$ with domain $\mathcal{D}(Q_q) = \mathcal{D}(\overline{\partial}_q) \cap \mathcal{D}(\overline{\partial}_{q-1}^*)$. The self-adjoint operator $\Box_q$ associated with $Q_q$ is called the $\overline{\partial}$-Neumann Laplacian on $L^2_{(0,q)}(\Omega)$. It is an elliptic operator with non-coercive boundary conditions [KN65].

The Dolbeault and $L^2$-cohomology groups on $\Omega$ are defined respectively by

$$H^{0,q}(\Omega) = \begin{cases} f \in C^\infty_{(0,q)}(\Omega) \mid \overline{\partial}_q f = 0 \\ \overline{\partial}_{q-1} g \in C^\infty_{(0,q-1)}(\Omega) \end{cases}$$

and

$$\tilde{H}^{0,q}(\Omega) = \begin{cases} f \in L^2_{(0,q)}(\Omega) \mid \overline{\partial}_q f = 0 \\ \overline{\partial}_{q-1} g \in L^2_{(0,q-1)}(\Omega) \end{cases}.$$  

These cohomology groups are in general not isomorphic. For example, when a complex variety is deleted from $\Omega$, the $L^2$-cohomology group remains the same but the Dolbeault cohomology group could change from trivial to infinite dimensional. As noted in the paragraph preceding Proposition 2.3, when $\mathcal{R}(\overline{\partial}_{q-1})$ is closed in $L^2_{(0,q)}(\Omega)$, $\tilde{H}^{0,q}(\Omega) \cong \mathcal{N}(\Box_q)$.

We refer the reader to [De] for an extensive treatise on the subject and to [H65] and [O82] for results relating these cohomology groups.

3. Positivity of the Spectrum and Essential Spectrum

Laufer proved in [L75] that for any open subset of a Stein manifold, if a Dolbeault cohomology group is finite dimensional, then it is trivial. In this section, we establish the following $L^2$-analogue of this result on a bounded domain in a Stein manifold:

**Theorem 3.1.** Let $\Omega \subset \subset X$ be a domain in a Stein manifold $X$ with $C^1$ boundary. Let $\Box_q$, $1 \leq q \leq n-1$, be the $\overline{\partial}$-Neumann Laplacian on $L^2_{(0,q)}(\Omega)$. Assume that $\mathcal{N}(\Box_q) \subset W^1(\Omega)$. Then $\inf \sigma(\Box_q) > 0$ if and only if $\inf \sigma_+(\Box_q) > 0$.
The proof of Theorem 3.1 follows the same line of arguments as Laufer’s. We provide the details below.

Let \( H^\infty(\Omega) \) be the space of bounded holomorphic functions on \( \Omega \). For any \( f \in H^\infty(\Omega) \), let \( M_f \) be the multiplication operator by \( f \):
\[
M_f : L^2_{(0,q)}(\Omega) \to L^2_{(0,q)}(\Omega), \quad M_f(u) = fu.
\]
Then \( M_f \) induces an endomorphism on \( \widetilde{H}^{0,q}(\Omega) \). Let \( I \) be set of all holomorphic functions \( f \in H^\infty(\Omega) \) such that \( M_f = 0 \) on \( \widetilde{H}^{0,q}(\Omega) \). Evidently, \( I \) is an ideal of \( H^\infty(\Omega) \). Assume \( \inf \sigma_e(\Box_q) > 0 \). To show that \( \widetilde{H}^{0,q}(\Omega) \) is trivial, it suffices to show that \( 1 \in I \).

**Lemma 3.2.** Let \( \xi \) be a holomorphic vector field on \( X \) and let \( f \in I \). Then \( \xi(f) \in I \).

**Proof.** Let \( D = \xi \partial : C^\infty_{(0,q)}(\Omega) \to C^\infty_{(0,q)}(\Omega) \), where \( \partial \) denotes the contraction operator. It is easy to check that \( D \) commute with the \( \overline{\partial} \) operator. Therefore, \( D \) induces an endomorphism on \( \widetilde{H}^{0,q}(\Omega) \). (Recall that under the assumption, \( \widetilde{H}^{0,q}(\Omega) \cong \mathcal{N}(\Box_q) \subset W^1(\Omega) \).) For any \( u \in \mathcal{N}(\Box_q) \),
\[
D(fu) - fD(u) = \xi \partial(fu) - f \xi \partial u = \xi(f)u.
\]
Notice that \( \Omega \) is locally starlike near the boundary. Using partition of unity and the Friedrichs Lemma, we obtain \( [D(fu)] = 0 \). Therefore, \( [\xi(f)u] = [D(fu)] - [fD(u)] = [0]. \)

We now return to the proof of the theorem. Let \( F = (f_1, \ldots, f_{n+1}) : X \to \mathbb{C}^{2n+1} \) be a proper embedding of \( X \) into \( \mathbb{C}^{2n+1} \) (cf. Theorem 5.3.9 in [H91]). Since \( \Omega \) is relatively compact in \( X \), \( f_j \in H^\infty(\Omega) \). For any \( f_j \), let \( P_j(\lambda) \) be the characteristic polynomial of \( M_{f_j} : \widetilde{H}^{0,q}(\Omega) \to \widetilde{H}^{0,q}(\Omega) \). By the Cayley-Hamilton theorem, \( P_j(M_{f_j}) = 0 \) (cf. Theorem 2.4.2 in [HJ85]). Thus \( P_j(f_j) \in I \).

The number of points in the set \( \{ (\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}) \in \mathbb{C}^{2n+1} \mid P_j(\lambda_j) = 0, 1 \leq j \leq 2n+1 \} \) is finite. Since \( F : X \to \mathbb{C}^{2n+1} \) is one-to-one, the number of common zeroes of \( P_j(f_j(z)) \), \( 1 \leq j \leq 2n+1 \), on \( X \) is also finite. Denote these zeroes by \( z^k, 1 \leq k \leq N \). For each \( z^k \), let \( g_k \) be a function in \( I \) whose vanishing order at \( z^k \) is minimal. (Since \( P_j(f_j) \in I \), \( g_k \neq 0 \).) We claim that \( g_k(z^k) \neq 0 \). Suppose otherwise \( g_k(z^k) = 0 \). Since there exists a holomorphic vector field \( \xi \) on \( X \) with any prescribed holomorphic tangent vector at any given point (cf. Corollary 5.6.3 in [H91]), one can find an appropriate choice of \( \xi \) so that \( \xi(g_j) \) vanishes to lower order at \( z^k \). According to Lemma 3.2, \( \xi(g_j) \in I \). We thus arrive at a contradiction.

Now we know that there are holomorphic functions, \( P_j(f_j), 1 \leq j \leq 2n+1 \), and \( g_k, 1 \leq k \leq N \), that have no common zeroes on \( X \). It then follows that there exist holomorphic functions \( h_j \) on \( X \) such that
\[
\sum P_j(f_j)h_j + \sum g_kh_k = 1.
\]
(See, for example, Corollary 16 on p. 244 in [GR65], Theorem 7.2.9 in [H91], and Theorem 7.2.5 in [Kr01]. Compare also Theorem 2 in [Sk72].) Since \( P_j(f_j) \in I, g_k \in I \), and \( h_j \in H^\infty(\Omega) \), we have \( 1 \in I \). We thus conclude the proof of Theorem 3.1.

**Remark.** (1) Unlike the above-mentioned result of Laufer on the Dolbeault cohomology groups [L75], Theorem 3.1 is not expected to hold if the boundedness condition on \( \Omega \) is removed (compare [W83]). It would be interesting to know whether Theorem 3.1 remains true if the assumption \( \mathcal{N}(\Box_q) \subset W^1(\Omega) \) is dropped and whether it remains true for unbounded pseudoconvex domains.
(2) Notice that in the above proof, we use the fact that $R(\partial q - 1)$ is closed, as a consequence of the assumption $\inf \sigma_e(\Box q) > 0$ by Proposition 2.3. It is well known that for any infinite dimensional Hilbert space $H$, there exists a subspace $R$ of $H$ such that $H/R$ is finite dimensional but $R$ is not closed. However, the construction of such a subspace usually involves Zorn’s lemma (equivalently, the axiom of choice). It would be of interest to know whether there exists a domain $\Omega$ in a Stein manifold such that $\tilde{H}^{0,q}(\Omega)$ is finite dimensional but $R(\partial q - 1)$ is not closed.

(3) We refer the reader to [Sh09] for related results on the relationship between triviality and finite dimensionality of the $L^2$-cohomology groups using the $\partial$-Cauchy problem. We also refer the reader to [B02] for a related result on embedded CR manifolds.

4. Hearing pseudoconvexity

The following theorem illustrates that one can easily determine pseudoconvexity from the spectrum of the $\bar{\partial}$-Neumann Laplacian.

**Theorem 4.1.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ such that $\text{int}(\text{cl}(\Omega)) = \Omega$. Then the following statements are equivalent:

1. $\Omega$ is pseudoconvex.
2. $\inf \sigma(\Box q) > 0$, for all $1 \leq q \leq n - 1$.
3. $\inf \sigma_e(\Box q) > 0$, for all $1 \leq q \leq n - 1$.

The implication (1) $\Rightarrow$ (2) is a consequence of Hörmander’s fundamental $L^2$-estimates of the $\bar{\partial}$-operator [H65], in light of Proposition 2.2 and it holds without the assumption $\text{int}(\text{cl}(\Omega)) = \Omega$. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are consequences of the sheaf cohomology theory dated back to Oka and Cartan (cf. Se53, L66, Sh67, Br83, O88). A elementary proof of (2) implying (1), as explained in [Fu05], is given below. The proof uses sheaf cohomology arguments in [L66]. When adapting Laufer’s method to study the $L^2$-cohomology groups, one encounters a difficulty: While the restriction to the complex hyperplane of the smooth function resulting from the sheaf cohomology arguments for the Dolbeault cohomology groups is well-defined, the restriction of the corresponding $L^2$ function is not. This difficulty was overcome in [Fu05] by appropriately modifying the construction of auxiliary $(0,q)$-forms (see the remark after the proof for more elaborations on this point).

We now show that (2) implies (1). Proving by contradiction, we assume that $\Omega$ is not pseudoconvex. Then there exists a domain $\tilde{\Omega} \supset \Omega$ such that every holomorphic function on $\Omega$ extends to $\tilde{\Omega}$. Since $\text{int}(\text{cl}(\Omega)) = \Omega$, $\tilde{\Omega} \setminus \text{cl}(\Omega)$ is non-empty. After a translation and a unitary transformation, we may assume that the origin is in $\tilde{\Omega} \setminus \text{cl}(\Omega)$ and there is a point $z^0$ in the intersection of the $z_n$-plane with $\tilde{\Omega}$ that is in the same connected component of the intersection of the $z_n$-plane with $\Omega$.

Let $m$ be a positive integer (to be specified later). Let $k_q = n$. For any $\{k_1, \ldots, k_{q-1}\} \subset \{1, 2, \ldots, n - 1\}$, we define

$$u(k_1, \ldots, k_q) = \frac{(q - 1)!((\bar{z}_{k_1} \cdots \bar{z}_{k_q})^{m-1}}{r_m^q} \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} \wedge \cdots \wedge d\bar{z}_{k_j} \wedge \cdots \wedge d\bar{z}_{k_q},$$

where $r_m = |z_1|^{2m} + \ldots + |z_n|^{2m}$. As usual, $d\bar{z}_{k_j}$ indicates the deletion of $d\bar{z}_{k_j}$ from the wedge product. Evidently, $u(k_1, \ldots, k_q) \in L^2_{(0,q-1)}(\Omega)$ is a smooth form on $\mathbb{C}^n \setminus \{0\}$. Moreover,
$u(k_1, \ldots, k_q)$ is skew-symmetric with respect to the indices $(k_1, \ldots, k_{q-1})$. In particular, $u(k_1, \ldots, k_q) = 0$ when two $k_j$'s are identical.

We now fix some notional conventions. Let $K = (k_1, \ldots, k_q)$ and $J$ a collection of indices from \{1, \ldots, k_q\}. Write $d\bar{z}_K = d\bar{z}_{k_1} \wedge \ldots \wedge d\bar{z}_{k_q}$, $\bar{z}_{K}^{m-1} = (\bar{z}_{k_1} \cdots \bar{z}_{k_q})^{m-1}$, and $d\bar{z}_K = d\bar{z}_{k_1} \wedge \ldots \wedge d\bar{z}_{k_q} \wedge \ldots \wedge d\bar{z}_{q}$. Denote by $(k_1, \ldots, k_q \setminus J)$ the tuple of remaining indices after deleting those in $J$ from $(k_1, \ldots, k_q)$. For example, $(2, 5, 3, 1 \setminus (4, 1, 6, 4, 6)) = (2, 5, 3)$.

It follows from a straightforward calculation that

$$
\bar{\partial}u(k_1, \ldots, k_q) = -\frac{q!mz_{K}^{m-1}}{r_{m+1}}(r_m d\bar{z}_K + \sum_{\ell=1}^{n} z_{\ell}^{m-1} z_{\ell}^{m} d\bar{z}_{\ell}) \wedge \left( \sum_{j=1}^{q} (-1)^j \bar{z}_{j} \bar{d}\bar{z}_{j} \right)
$$

$$
= -\frac{q!mz_{K}^{m-1}}{r_{m+1}} \sum_{\ell=1}^{n} \sum_{\ell \neq k_1, \ldots, k_q} z_{\ell}^{m} z_{\ell}^{m-1} \left( \bar{z}_{\ell} d\bar{z}_K + d\bar{z}_{\ell} \wedge \sum_{j=1}^{q} (-1)^j \bar{z}_{j} \bar{d}\bar{z}_{j} \right)
$$

$$
= m \sum_{\ell=1}^{n-1} z_{\ell}^{m} u(\ell, k_1, \ldots, k_q).
$$

(4.2)

It follows that $u(1, \ldots, n)$ is a $\bar{\partial}$-closed $(0, n-1)$-form.

By Proposition 2.2, we have $\mathcal{R}(\bar{\partial}q) = \mathcal{N}(\bar{\partial}q)$ for all $1 \leq q \leq n-1$. We now solve the $\bar{\partial}$-equations inductively, using $u(1, \ldots, n)$ as initial data. Let $v \in L^2_{(0,n-2)}(\Omega)$ be a solution to $\bar{\partial}v = u(1, \ldots, n)$. For any $k_1 \in \{1, \ldots, n-1\}$, define

$$w(k_1) = -mz_{k_1}^{m} v + (-1)^{1+k_1} u(1, \ldots, n \mid k_1).
$$

Then it follows from (4.2) that $\bar{\partial}w(k_1) = 0$. Let $v(k_1) \in L^2_{(0,n-3)}(\Omega)$ be a solution of $\bar{\partial}v(k_1) = w(k_1)$.

Suppose for any $(q-1)$-tuple $K' = (k_1, \ldots, k_{q-1})$ of integers from \{1, \ldots, n-1\}, $q \geq 2$, we have constructed $v(K') \in L^2_{(0,n-q-1)}(\Omega)$ such that it is skew-symmetric with respect to the indices and satisfies

$$
\bar{\partial}v(K') = m \sum_{j=1}^{q-1} (-1)^j z_{k_j}^{m} v(K' \mid k_j) + (-1)^{q+|K'|} u(1, \ldots, n \mid K')
$$

(4.3)

where $|K'| = k_1 + \ldots + k_{q-1}$ as usual. We now construct a $(0, n-q-2)$-forms $v(K)$ satisfying (4.3) for any $q$-tuple $K = (k_1, \ldots, k_q)$ of integers from \{1, \ldots, n-1\} (with $K'$ replaced by $K$). Let

$$w(K) = m \sum_{j=1}^{q} (-1)^j z_{k_j}^{m} v(K \mid k_j) + (-1)^{q+|K'|} u(1, \ldots, n \mid K).
$$
Then it follows from (4.2) that

$$\overline{\partial} w(K) = m \sum_{j=1}^{q} (-1)^j z_k^m \overline{\partial}_v(K | k_j) + (-1)^{q+|K|} \overline{\partial}_u(1, \ldots, n | K)$$

$$= m \sum_{j=1}^{q} (-1)^j z_k^m \left( m \sum_{1 \leq i < j} (-1)^i z_k^m v(K | k_j, k_i) + m \sum_{j < i \leq q} (-1)^{i-1} z_k^m v(K | k_j, k_i) \right)$$

$$- (-1)^{q+|K|} \overline{\partial}_u(1, \ldots, n | (K | k_j)) + (-1)^{q+|K|} \overline{\partial}_u(1, \ldots, n | K)$$

$$= (-1)^{q+|K|} (- m \sum_{j=1}^{q} (-1)^j z_k^m u(k_j, (1, \ldots, n | K)) + \overline{\partial}_u(1, \ldots, n | K)) = 0$$

Therefore, by the hypothesis, there exists a $v(K) \in L^2_{[0,n-q-2]}(\Omega)$ such that $\overline{\partial} v(K) = w(K)$. Since $w(K)$ is skew-symmetric with respect to indices $K$, we may also choose a likewise $v(K)$. This then concludes the inductive step.

Now let

$$F = w(1, \ldots, n-1) = m \sum_{j=1}^{n-1} z_j^m v(1, \ldots, j, \ldots, n-1) - (-1)^{n+\frac{n(n-1)}{2}} u(n),$$

where $u(n) = -z_n^m/t_m$, as given by (4.1). Then $F(z) \in L^2(\Omega)$ and $\overline{\partial} F(z) = 0$. By the hypothesis, $F(z)$ has a holomorphic extension to $\overline{\Omega}$. We now restrict $F(z)$ to the coordinate hyperplane $z' = \{z_1, \ldots, z_{n-1} = 0\} = 0$. Notice that so far we only choose the $v(K)$'s and $w(K)$'s from $L^2$-spaces. The restriction to the coordinate hyperplane $z' = 0$ is not well-defined. To overcome this difficulty, we choose $m > 2(n-1)$. For sufficiently small $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ \int_{\{|z'| < \varepsilon\} \cap \Omega} \left| (F + (-1)^{n+\frac{n(n-1)}{2}} u(n))(\delta z', z_n) \right|^2 dV(z) \right\}^{1/2}$$

$$\leq m \delta^{m+\frac{n(n-1)}{2}} \sum_{j=1}^{n-1} \left\{ \int_{\{|z'| < \varepsilon\} \cap \Omega} |v(1, \ldots, j, \ldots, n-1)(\delta z', z_n)|^2 dV(z) \right\}^{1/2}$$

$$\leq m \delta^{m-2(n-1)} \varepsilon^{m} \sum_{j=1}^{n-1} \|v(1, \ldots, j, \ldots, n-1)\|_{L^2(\Omega)}.$$

Letting $\delta \to 0$, we then obtain

$$F(0, z_n) = -(-1)^{n+\frac{n(n-1)}{2}} u(n)(0, z_n) = (-1)^{n+\frac{n(n-1)}{2}} z_n^{-m}.$$ for $z_n$ near $z_0^{0}$. (Recall that $z_0 \in \Omega$ is in the same connected component of $\{z' = 0\} \cap \overline{\Omega}$ as the origin.) This contradicts the analyticity of $F$ near the origin. We therefore conclude the proof of Theorem (1).

Remark. (1) The above proof of the implication (2) $\Rightarrow$ (1) uses only the fact that the $L^2$-cohomology groups $\check{H}^{0,q}(\Omega)$ are trivial for all $1 \leq q \leq n-1$. Under the (possibly) stronger assumption $\inf \sigma(\square_q) > 0$, $1 \leq q \leq n-1$, the difficulty regarding the restriction of the
$L^2$ function to the complex hyperplane in the proof becomes superficial. In this case, the \( \bar{\partial} \)-Neumann Laplacian \( \square_q \) has a bounded inverse. The interior ellipticity of the \( \bar{\partial} \)-complex implies that one can in fact choose the forms \( v(K) \) and \( w(K) \) to be smooth inside \( \Omega \), using the canonical solution operator to the \( \bar{\partial} \)-equation. Therefore, in this case, the restriction to \( \{z' = 0\} \cap \Omega \) is well-defined. Hence one can choose \( m = 1 \). This was indeed the choice in \[L66\], where the forms involved are smooth and the restriction posts no problem. It is interesting to note that by having the freedom to choose \( m \) sufficiently large, one can leave out the use of interior ellipticity. Also, the freedom to choose \( m \) becomes crucial when one proves an analogue of Theorem 4.1 for the Kohn Laplacian because the \( \bar{\partial}_b \)-complex is no longer elliptic. The construction of \( u(k_1, \ldots, k_q) \) in (4.1) with the exponent \( m \) was introduced in \[Fu05\] to handle this difficulty.

(2) One can similarly give a proof of the implication \((3) \Rightarrow (1)\). Indeed, the above proof can be easily modified to show that the finite dimensionality of \( \tilde{H}^{0,q}(\Omega) \), \( 1 \leq q \leq n - 1 \), implies the pseudoconvexity of \( \Omega \). In this case, the \( u(K) \)'s are defined by

\[
\begin{align*}
 u(k_1, \ldots, k_q) &= \frac{(\alpha + q - 1)!z^\alpha_n(z_{k_1} \cdots z_{k_q})^{m-1}}{r_m^{\alpha+q}} \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} d\bar{z}_{k_1} \wedge \cdots \wedge \hat{d}\bar{z}_{k_j} \wedge \cdots \wedge d\bar{z}_{k_q},
\end{align*}
\]

where \( \alpha \) is any non-negative integers. One now fixes a choice of \( m > 2(n-1) \) and let \( \alpha \) runs from 0 to \( N \) for a sufficiently large \( N \), depending on the dimensions of the \( L^2 \)-cohomology groups. We refer the reader to \[Fu05\] for details.

(3) As noted in Sections 2 and 3, unlike the Dolbeault cohomology case, one cannot remove the assumption \( \text{int}(\overline{\text{cl}(\Omega)}) = \Omega \) or the boundedness condition on \( \Omega \) from Theorem 4.1. For example, a bounded pseudoconvex domain in \( \mathbb{C}^n \) with a complex analytic variety removed still satisfies condition (2) in Theorem 3.1.

(4) As in \[L66\], Theorem 4.1 remains true for a Stein manifold. More generally, as a consequence of Andreotti-Grauert’s theory \[AG62\], the \( q \)-convexity of a bounded domain \( \Omega \) in a Stein manifold such that \( \text{int}(\overline{\text{cl}(\Omega)}) = \Omega \) is characterized by \( \inf \sigma(\square_k) > 0 \) or \( \inf \sigma_e(\square_k) > 0 \) for all \( k \leq n-1 \).

(5) It follows from Theorem 3.1 in \[H04\] that for a domain \( \Omega \) in a complex hermitian manifold of dimension \( n \), if \( \inf \sigma_e(\square_q) > 0 \) for some \( q \) between 1 and \( n-1 \), then wherever the boundary is \( C^3 \)-smooth, its Levi-form cannot have exactly \( n-q-1 \) positive and \( q \) negative eigenvalues. A complete characterization of a domain in a complex hermitian manifold, in fact, even in \( \mathbb{C}^n \), that has \( \inf \sigma_e(\square_q) > 0 \) or \( \inf \sigma(\square_q) > 0 \) is unknown.

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